Kalman-like filtering in a qualitative setting
A preliminary draft

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Abstract
A qualitative counterpart of Kalman filtering is proposed, which is contrasted with updating operations based on imaging in the sense of Lewis. Filtering and updating are defined and compared in the framework of possibility theory. A syntactic counterpart of qualitative filtering (and updating) is outlined in the setting of possibilistic logic.

1. Introduction

Since the pioneering work of Alchourron, Gärdenfors and Makinson (1985), and the publication of the seminal book on "Knowledge in Flux" by Gärdenfors (1988), there has been an important and increasing interest in the modelling of belief change. Progressively, basic distinctions have been emerged between various types of belief change: revision of beliefs by an input information in a static world vs. update of beliefs in a dynamic world (Katsuno and Mendelzon, 1991), revision by an input held as certain and prioritary vs. revision by an uncertain information (Darwiche and Pearl, 1997; Boutilier et al., 1998; Dubois and Prade, 1997a), revision vs. focusing on a class of reference (Dubois and Prade, 1997b), revision of beliefs vs. revision of preferences (Ryan and Williams, 1997; Benferhat et al., 1999). What is noticeable is that these distinctions can be made in various representational settings provided that these frameworks, which might be symbolic or numerical (Léa Sombé, 1994; Dubois and Prade, 1998b), are rich enough for enabling the expression of these distinctions.

Another important aspect with respect to belief revision is the epistemic entrenchment underlying any well-behaved revision process, which should obey Alchourron, Gärdenfors and Makinson (AGM) postulates. Since an epistemic entrenchment relation is closely related to a necessity measure in the sense of possibility theory (Dubois and Prade, 1991), the framework of possibilistic logic (Dubois, Lang, Prade, 1994) enables us to envisage belief revision both at the syntactic level of a possibilistic logic base, and in an equivalent manner, at the semantic level of a possibility distribution ranking the interpretations. In this approach the ordering on which the revision is based is explicitly associated with the formulas and is revised in the revision process. This view is also advocated by Williams (1994) in her related approach based on adjustments.

The present paper should be understood in this general perspective, where different types of belief change operations have been investigated both at the semantic and at the syntactic level. A qualitative counterpart of a well-known "updating" method, Kalman filtering (briefly recalled in Section 2), is introduced in Section 3 and compared to updating based on imaging in Sections 4 and 5, in the setting of possibility theory. Then a syntactic counterpart of these machineries is outlined in Section 6.

2. Kalman filtering

Kalman filtering is the basis of well-known updating techniques in systems engineering (e.g., Bar-Shalom and Fortmann, 1988), in the case of an evolving system when events are dated. The idea underlying Kalman filtering, namely a two-steps procedure involving prediction and revision, can be of interest in other settings. Recently, Castel, Cossart and Tessier (1998), Cossart and Tessier (1999) have proposed to transpose these ideas in a symbolic setting for a situation assessment problem. Let us first consider the probabilistic framework.

We only give here the basic principles on which Kalman filtering is based. Let Ω be a set of possible worlds. It is assumed that there exists a prediction function f such that $f(\omega_t) = \omega_{t+1}$, where $\omega_t \in \Omega$ is the state at time t of a consistent system and $f(\omega_t)$ is the resulting state at time t + 1. Knowing the probability distribution p_t on the system state at time t, the prevision (forecast distribution) at t + 1 is given in ω by $P_t(f^{-1}(\omega))$ where $f^{-1}(\omega) = \{\omega' : \omega = f(\omega')\}$. Let A be an input information available at time t+1, the updated state at time t + 1 could be computed using Bayes rule as

$$p_{t+1}(\omega) = P_t(f^{-1}(\omega) \mid A) = P_t(f^{-1}(\omega) \cap A) \,/\, P_t(A). \qquad (1)$$

Thus this type of updating is decomposed into a prediction step followed by a revision step. The underlying idea is that the prediction of the next step at t+1 pervaded with uncertainty is improved by taking into account the observation A.

3. Possibilistic filtering

A brief background on possibility theory is first given in a belief change perspective, before proposing a possibilistic counterpart of (1).

3.1. The possibility theory setting

Possibility theory provides a framework for uncertainty modelling, which can be numerical or remain qualitative, and which departs from probability by the use of maxitive (rather than additive) law and the existence of a dual pair of measures for assessing the uncertainty. See Dubois and Prade (1998a) for a detailed overview of possibility theory. The possibilistic approach enriches the knowledge representation provided by the pure logical setting from the point of view of expressiveness. Instead of viewing a belief state as a flat set Ω of mutually exclusive states, one adds a complete partial ordering on top, according to which some states are considered as more plausible than others (Dubois and Prade, 1988; Zadeh, 1978). A cognitive state can then be modelled by a possibility distribution π, that is, a mapping from Ω to a totally ordered set V containing a greatest element (denoted 1) and a least element (denoted 0), typically the unit interval V = [0,1]. However any finite, or infinite and bounded, chain will do as well. This approach is also close to Spohn (1988)'s well-ordered partitions, see (Dubois and Prade, 1991).

A consistent cognitive state π is such that $\pi(\omega) = 1$ for some ω, i.e., at least one of the states is considered as completely possible in Ω. In such a case π is said to be normalized. Here consistency can be a matter of degree. A cognitive state π is said to be partially inconsistent if $0 < \max \{\pi(\omega) : \omega \in \Omega\} < 1$. When $\pi(\omega) > \pi(\omega')$ then ω is a more plausible state than ω'. When there is one state $\omega_0 \in \Omega$ such that $\pi(\omega_0) = 1$, and $\pi(\omega) = 0$ if $\omega \neq \omega_0$, π corresponds to a *complete* cognitive state. Conversely, the vacuous cognitive state is expressed by the least specific possibility distribution on Ω, i.e., $\pi_T(\omega) = 1, \forall \omega$. It corresponds to the state of total ignorance. A possibility measure Π is associated with a possibility distribution π, namely:

$$\Pi(A) = \sup_{\omega \in \Omega} \pi(\omega).$$

Possibility measures thus satisfy the following characteristic decomposition property:

$$\Pi(A \cup B) = \max(\Pi(A), \Pi(B)).$$

Necessity measures N are defined by duality, namely

$$N(A) = 1 - \Pi(\neg A)$$

and $N(A \cap B) = \min(N(A), N(B))$.

3.2. Possibilistic revision

The three basic forms of belief dynamics described by Gärdenfors (1988), namely expansion, contraction and revision can easily be depicted in the possibilistic framework. Only expansion and revision are now recalled. The expansion π^+_A of a cognitive state π upon learning the sure fact A makes full sense when A is fully consistent with the prior cognitive state described by π, that is, if $\exists\, \omega \in A, \pi(\omega) = 1$. The expansion π^+_A in the consistent case is defined as:

$$\forall \omega,\ \pi^+_A(\omega) = \min(\mu_A(\omega), \pi(\omega)) \qquad (2)$$

where μ_A is the characteristic function of the subset A. If the input A is not fully consistent with π ($\forall \omega \in A, \pi(\omega)<1$) then one considers that expansion yields an absurd belief state: $\pi^+_A(\omega) = 0, \forall \omega$. The result of an expansion, that stems from receiving new information consistent with a previously available cognitive state described by π, is another possibility distribution π^+_A that is more restrictive ($\pi^+_A \leq \pi$), and thus more informative than π.

In a case of inconsistency of the input with the cognitive state, a revision process takes place. It consists in transforming the cognitive state π into a possibility distribution π^*_A obtained by revising π with input A. This new possibility distribution is obtained by letting $\pi^*_A = \pi(\cdot \mid A)$ where $\pi(\cdot \mid A)$ denotes the qualitative possibilistic conditioning defined by

$$\begin{aligned} \pi(\omega \mid A) &= 1 \text{ if } \pi(\omega) = \Pi(A), \omega \in A \\ &= \pi(\omega) \text{ if } \pi(\omega) < \Pi(A), \omega \in A \qquad (3) \\ &= 0 \text{ if } \omega \notin A. \end{aligned}$$

It has been shown (Dubois and Prade, 1992) that when defining π^*_A as $\pi(\cdot \mid A)$ the associated revision process * satisfies all AGM postulates which underlies any well-behaved belief revision (Gärdenfors, 1988). Note that (2) only requires an ordinal scale. In a numerical setting conditioning may be defined by:

$$\pi(\omega \mid A) = \frac{\pi(\omega)}{\Pi(A)}, \; \forall\, \omega \in A \qquad (4)$$
$$= 0 \text{ otherwise,}$$

which is a particular case of Dempster rule of conditioning (Shafer, 1976).

3.3. Filtering in the possibilistic framework

Let us now give the possibilistic counterpart of Kalman filtering in the sense of (1). Other counterparts of the Kalman filtering ideas have been recently proposed in the possibility theory framework by Delplanque et al. (1997) in a problem of underwater robotics, and by Nifle and Reynaud (1997) for the recognition of fuzzily described temporal scenarios.

Let f be a prediction function $f(\omega_t) = \omega_{t+1}$, where ω_t is the state at time t. Knowing the possibility distribution π_t on the system state at time t, the prevision (forecast distribution) at t + 1 is given in ω by $\Pi_t(f^{-1}(\omega))$, where $\Pi_t(f^{-1}(\omega)) = \max \{\pi_t(\omega') : \omega' \in f^{-1}(\omega)\}$, and $\Pi_t(\varnothing) = 0$. Let A be an input information available at time t + 1, the updated state at time t + 1 could be computed using the possibilistic revision process defined above (2) namely:

$$\pi_{t+1}(\omega) = \Pi_t(f^{-1}(\omega) \mid A) = \max_{\omega' \in f^{-1}(\omega)} \pi_t(\omega' \mid A) \,. \qquad (5)$$

Note that π_{t+1} is always normalized (if π_t is). In the above formula, it would be possible to replace $\pi_t(\omega' \mid A)$ by a more general expression in case of an uncertain observation (A, α). See Dubois and Prade (1997a) for conditioning by an uncertain input.

More generally, one may consider a family $\{\pi_\omega, \omega \in \Omega\}$ describing a transition graph, hence generalizing f as a fuzzy relation R, such that $\mu_R(\omega,\omega') = \pi_\omega(\omega')$ is the plausibility that ω' follows ω, and then compute the image of the cognitive state pertaining to the initial state through the fuzzy relation R (prediction) and revise the so-obtained prediction by the input, that is, in the timed setting, compute the updated possibility distribution π_{t+1}

$$\pi_{t+1}(\omega') = \max_\omega \min(\pi_t(\omega \mid A), \pi_\omega(\omega')) \,. \qquad (6)$$

Note that π_{t+1} is normalized provided that $\exists \omega, \omega'$ such that $\pi_t(\omega)=1$ and $\pi_\omega(\omega')=1$.

4 - Updating

In the following, updating precisely refers to the befief change operation which aims at restoring uptodate views of the world in a dynamic world when receiving new information. At the theoretical level probabilistic imaging belongs to this type of operation. We then consider its possibilistic counterpart.

4.1 Probabilistic imaging

Another path in the problem of probabilistic change, which departs both from conditioning and filtering, is the one followed by Lewis (1976). Assume that the set Ω of possible states possesses a distance measure and is such that for any state $\omega \in \Omega$, and any set $A \subseteq \Omega$, there is a single state ω_A in A defined as the closest state to ω. Then the principle of minimal change upon learning that some event $A \subseteq \Omega$ has occurred can be expressed as an advice to allocate the probability weight of each state that becomes impossible to the closest state that is made possible by the input. The input is here at the same level of generality as the prior probability, and the translation of worlds expresses that the current state has changed, and not that our previous beliefs about it were wrong. This updating rule can be formally expressed as

$$\forall\ \omega' \in A,\ p_A(\omega') = \Sigma_{\omega:\ \omega'=\omega_A}\ p(\omega). \qquad (7)$$

This rule is called 'imaging' because p_A is the image of p on A obtained by moving the masses $p(\omega)$ for $\omega \notin A$ to $\omega_A \in A$, with the natural convention that $\omega_A = \omega$ if $\omega \in A$. This rule actually comes from the study of conditional logics (Harper et al., 1981), and was motivated by the study of the probability of a conditional in such logics. It turned out that computing such a probability led to imaging and not to the usual conditional probability.

The imaging rule has been generalized by Gärdenfors (1988) to the case when the set of states in A closest to a given state ω contains more than one element. If $A(\omega) \subseteq A$ is the subset of closest states from ω, $p(\omega)$ can be shared among the various states $\omega' \in A(\omega)$ instead of being allocated to a unique state. Clearly, instead of sharing $p(\omega)$ among $\omega' \in A(\omega)$, a less committed update is to allocate $p(\omega)$ to $A(\omega)$ itself (and none of its subsets). In that case the imaging process produces a basic probability assignment (Shafer, 1976) in the sense of Dempster (1967)'s view of belief functions. But this type of update is not consistent with Bayesian probabilities because the result of imaging is a family of probability distributions, and not a unique one.

Note that imaging can turn impossible states into possible ones, i.e., one may have $p_A(\omega) > 0$ while $p(\omega) = 0$ for some ω, e.g., if ω_A is such that $p(\omega_A) = 0$. As a consequence a sure fact B a priori, i.e., such that $P(B) = 1$ may become uncertain, i.e., $P_A(B) < 1$. This is not the case with Bayesian conditioning. In order to preserve this kind of monotonicity property, one idea (see Gärdenfors, 1988) is to build P_A as the image of P on $A \cap S$ where $S = \{\omega \mid P(\omega) > 0\}$ is the support of P. However, as with the Bayesian rule, $P(A) = 1 \Rightarrow P_A = P$; this is the probabilistic version of the success postulate of Katsuno and Mendelzon (1991) for updating. In fact, all postulates of Katsuno and Mendelzon hold or have a natural counterpart for probabilistic cognitive states, except the postulate which expresses that the conjunction of B with the result of an updating by A entails the result of the updating by the conjunction of A and B (see, e.g., Léa Sombé, 1994).

4.2 Possibilistic imaging

It is easy to define the possibilistic counterpart to Lewis' imaging since this type of belief change is based on mapping each possible state to the closest one that accommodates the input information. As above, define for any $\omega \in \Omega$, and non-empty set $A \subseteq \Omega$ the closest state to ω where A is true, that is, where $\omega_A \in A$. Then the image $\pi°_A$ of a cognitive state π in A is such that

$$\pi°_A(\omega') = \max_{\omega:\ \omega'=\omega_A} \pi(\omega) \text{ if } \omega' \in A$$
$$= 0 \text{ if } \omega' \notin A. \quad (8)$$

If there is more than one state ω_A closest to ω, then the weight $\pi(\omega)$ can be allocated to each of the closest states forming the set $A(\omega)$, and the above imaging rule becomes

$$\pi°_A(\omega') = \max_{\omega:\ \omega'\in A(\omega)} \pi(\omega) \qquad \text{if } \omega' \in A$$
$$= 0 \text{ if } \omega' \notin A. \quad (9)$$

Note that $\pi°_A$ is normalized if π is normalized. Defining $\forall\omega$, $A(\omega)$ precisely as $\{\omega' \mid \pi(\omega') = \Pi(A)\}$, which does not depend on ω, then $\pi°_A = \pi(\cdot \mid A)$, i.e., we recover the revision based on conditioning. Clearly in this setting, we see that possibilistic imaging formally subsumes the AGM revision. However this link is somewhat artificial. Indeed imaging can be envisaged in a dynamic perspective in which $A(\omega)$ represents the states where A is true that most plausibly follow ω. Clearly A(ω) depends on the current system state ω. Then input A warns the agent that a change in that system state has occurred.

It is easy to check that the above updating rule defined by (8) satisfies all postulates of Katsuno and Mendelzon (1991)'s updates (see Dubois, Moral and Prade, 1998). Katsuno and Mendelzon (1991) have proved that any change operation that obeys all postulates involves a proximity structure on Ω, that is, a family $\{<_\omega, \omega \in \Omega\}$ of partial ordering relations, where $\omega'' <_\omega \omega'$ means that ω'' is closer than ω' to ω. In a dynamic system perspective, a state is the state of a dynamic system and $\{<_\omega, \omega \in \Omega\}$ represents a partial transition graph where $\omega'' <_\omega \omega'$ means that ω'' is a more plausible successor to ω than ω'. Then A(ω) gathers all states in A that are minimal in the sense of $<_\omega$.

It has been shown in (Dubois, Dupin and Prade, 1995) that adding one more postulate the proximity structure on Ω is a family $\{\leq_\omega, \omega \in \Omega\}$ of *complete* preordering relations, that can be equivalently represented by a family $\{\pi_\omega, \omega \in \Omega\}$ of qualitative possibility distributions. Then the most plausible states in A reachable from ω form the set $A(\omega) = \{\omega' \in A, \pi_\omega(\omega') = \Pi_\omega(A)\}$ where Π_ω is the possibility measure associated to π_ω. Defining R_A as the relation that to each ω assigns its closest neighbours A(ω) in A, the above update formula (8) is nothing but Zadeh (1965)'s extension principle that characterizes the fuzzy image of the fuzzy set whose membership function is π. Namely, if $\pi = \mu_F$ then

$$\pi^\circ{}_A = \mu_{R_A \hat{o} F}$$

with $\mu_{R_A}(\omega,\omega') = 1$ if $\omega' \in A(\omega)$ and $\mu_{R_A}(\omega,\omega') = 0$ otherwise and $\mu_{R_A \hat{o} F}(\omega') = \max_\omega \min(\pi(\omega), \mu_{R_A}(\omega,\omega'))$. In other words, the uncertainty on the initial system state is propagated over to the next state via the input-dependent prediction relation based on the transition graph.

More generally, using a family $\{\pi_\omega, \omega \in \Omega\}$ describing a transition graph, we may compute the image of the cognitive state pertaining to the initial state through the fuzzy relation $\{\pi_\omega, \omega \in \Omega\}$ (prediction) and revise the so-obtained prediction by the input, that is, in the timed setting, compute the updated possibility distribution π_{t+1}

$$\hat{e}\pi_{t+1}(\omega') = \max_\omega \min(\pi_t(\omega), \pi_\omega(\omega')) \text{ (prevision);}$$

$$\pi_{t+1}(\omega') = \hat{e}\pi_{t+1}(\omega' \mid A) \text{ (revision).}$$

This can be viewed as a generalized form of update. Note that $\hat{e}\pi_{t+1}$ is normalized provided that $\exists\omega$, ω' such that $\pi_t(\omega)=1$ and $\pi_\omega(\omega')=1$. However, if ω and ω' are such that $\pi(\omega) = 1$ and $\pi_\omega(\omega') = \prod_\omega(A)$ then $\pi°_A(\omega') = 1$ while $\pi_{t+1}(\omega') < 1$ if there exists $\omega'' \in A$ such that $\hat{e}\pi_{t+1}(\omega'') > \hat{e}\pi_{t+1}(\omega')$. This situation occurs if the transition to ω'' (from a highly plausible state different from ω) is more plausible than the transition from ω to ω'. This type of update operation can be encountered in other settings (Cordier and Siegel, 1993), (Cordier and Lang, 1995).

5. Filtering vs. imaging

We first highlight the differences between imaging and Kalman filtering, and then we show how formally imaging can be encoded as a Kalman-like filtering.

5.1. Basic differences

Clearly, filtering and imaging use equations presenting some similarities in order to compute the new cognitive state after learning some new event A. However there are several differences. In Kalman filtering, any prediction function can be used, and it does not depend on the event A. However, in imaging the distance is a strong constraint since if $\omega \in A$ then the closest interpretation of ω in A is ω itself.

Moreover, in imaging no possible initial state in A ($p(\omega) > 0$ and $\omega \in A$) is deemed impossible after A has occurred, since the revised prediction function f_A depends on A and is such that $f_A(\omega) = \omega_A \in A$. Imaging thus comes down in the probabilistic setting to computing $p_A(\omega) = P_t(f_A^{-1}(\omega))$ for all $\omega \in A$, and does not require any normalization since $P_A(A) = 1$. Moreover, in imaging, we always have: $\forall\omega \notin A$, $\pi°_A(\omega) = 0$. This is not necessarily true using filtering.

While $A(\omega)$ is a subset of A in the imaging, the value of the prediction function, and more generally π_ω, does not depend on A when filtering. Instead of selecting $A(\omega)$, generalized filtering considers the family $\{\pi_\omega, \omega \in \Omega\}$ describing the transition graph, as a fuzzy relation R such that $\mu_R(\omega,\omega') = \pi_\omega(\omega')$.

It is clear that π_{t+1} in (4) differs from $\pi°_A$ in (8) because they correspond to different strategies. Using $\pi°_A$ the assumed transition from each state ω is always supposed to be the most plausible one(s) gathered in $A(\omega)$, and the intrinsic plausibility of this transition is not considered. Using π_{t+1}, transitions that are not the most plausible ones compatible with A are considered via π_ω and lead to possible final states that are neglected by imaging. Hence the two approaches are different. However it is obvious that imaging makes sense for answering questions about the next most plausible state, while the prediction/revision approach is more adapted to the handling of trajectories in the transition graph, and is the counterpart in the possibilistic setting of Kalman filtering.

5.2. Imaging as a particular case of filtering

This subsection explains how imaging can be encoded using filtering. It is based on the following three remarks:

- There is no restriction on the function f in Kalman-like filtering. Hence, a distance measure used in imaging can be encoded using some particular kind of functions. Indeed, let A be a subset of Ω and let some distance d which gives for each interpretation ω its closest interpretation ω' in A. Then for each A, and for each d, we define $f_{A,d}$ in the following way:

$$\forall \omega,\ f_{A,d}(\omega) = \omega' \text{ where } \omega' \text{ is the closest interpretation to } \omega \text{ in A.} \qquad (10)$$

- As it is said above, in imaging, we always have $\forall \omega \notin A,\ \pi°_A(\omega) = 0$.

This is in general not true in the filtering framework. However, using equation (10), we can easily check that:

$$\forall \omega \notin A,\ \prod_t(f^{-1}_{A,d}(\omega)) = 0$$

since $\forall \omega \notin A,\ f^{-1}_{A,d}(\omega) = \emptyset$, namely, from (10),for any ω, $f_{A,d}(\omega)$ always belong to A.

- Now in filtering, when some event B is observed at time t+1, then all the possibility degrees $\pi_t(\omega)$ where $\omega \notin B$ are not taken into account in the computation of π_{t+1}. This is due to the fact that π_{t+1} uses the conditioning on π_t. However, in imaging all the degrees in π_t are used independently if ω belongs or not to the new information. Therefore, in order to recover imaging using filtering, we should take B as a tautology.

On the basis of the above points, the following proposition show formally how to recover imaging using Kalman filtering:

Proposition**:** Let A be a subset of Ω, π be a possibility distribution. Let $f_{A,d}$ be the prevision function as defined in (10), given some distance d. Then we have:

$\forall\omega,\quad \pi^{\circ}{}_{A}(\omega) = \pi_{t+1}(\omega)$

where $\pi_{t+1}(\omega) = \prod(f_{A,d}{}^{-1}(\omega) \mid \top)$ and $\top$ is a tautology.

The converse does not hold. This is mainly due to the strong assumption imposed by the distance where if $\omega \in A$ then the closest interpretation of ω in A is ω itself. Then, assume that we have a prediction function f and a formula A different from a tautology. Let B a formula where its models are the set of interpretations ω such that $\pi_{t+1}(\omega) > 0$ (B is necessarily different from tautology). In order to recover filtering using imaging, one should let B as the input in the imaging. Now, the above proposition does not hold because f should satisfy $\forall\omega \in B$, $f_A(\omega) = \omega$, which in general does not hold.

6. Syntactic filtering

Filtering (and updating) have been defined at the semantic level. In this section we provide their syntactic counterparts. We first give a compact representation of a possibility distribution by means of possibilistic knowledge bases.

6.1. Background on possibilistic logic

A possibilistic knowledge base is made up of a finite set of weighted formulas

$$\Sigma=\{(\phi_i,\ a_i),\ i=1,n\}$$

where a_i is understood as a lower bound on the degree of necessity $N(\phi_i)$. Formulas with zero degree are not explicitly represented in the knowledge base (only beliefs which are somewhat accepted by the agent are explicitly represented). The higher the weight, the more certain the formula. The weights a_i are hence viewed as constraints on possibility distributions. Indeed, each pair $(\phi_i,\ a_i)$ imposes that the induced possibility distribution π should satisfy: $1 - \max\{\pi(\omega) : \omega \,{}^{\circ}\phi_i\} \geq a_i$. Let $\Sigma_{\geq a_i}$ be the set of formulas with weight at least equal to a_i.

A possibilistic knowledge base Σ is said be consistent if its classical knowledge base, obtained by forgetting the weights, is classically consistent. We denote by

$$Inc(\Sigma) = \max\{a_i : \Sigma_{\geq a_i} \text{ is inconsistent}\}$$

the inconsistency degree of Σ. $Inc(\Sigma) = 0$ means that $\Sigma_{\geq ai}$ is consistent for all a_i.

Given a possibilistic knowledge base ∑, we can generate a unique possibility distribution by associating to each interpretation, the level of compatibility with agent's beliefs, i.e., with ∑. When a possibilistic knowledge base only consists of one formula $\{(\phi, a)\}$, then each interpretation ω which satisfies φ will have the possibility degree $\pi(\omega) = 1$ since it is consistent with φ, and each interpretation ω which falsifies φ will have a possibility degree $\pi(\omega)$ such that the higher a is (i.e., the more certain φ is), the lower $\pi(\omega)$ is. In particular, if $a = 1$ (i.e., φ is completely certain), then $\pi(\omega) = 0$, namely ω is impossible. One way to realize this constraint is to assign to $\pi(\omega)$ the degree $1 - a$ with a numerical encoding. Therefore, the possibility distribution associated with $\sum=\{(\phi, a)\}$ is:

$$\forall \omega \in \Omega,\ \pi_{\{(\phi\ a)\}}(\omega) = 1 \quad \text{if } \omega \in [\phi]$$
$$= 1 - a \quad \text{otherwise.}$$

where [φ] denotes the models of φ. When $\sum = \{(\phi_i, a_i), i=1,n\}$ is a general possibilistic knowledge base then all the interpretations satisfying all the beliefs in ∑ will have the highest possibility degree, namely 1, and the other interpretations will be ranked w.r.t. the highest belief that they falsify, namely we get (Dubois et al., 1994):

The possibility distribution associated with a knowledge base ∑ is defined by:

$$\forall \omega \in \Omega,\ \pi_{\sum}(\omega) = 1 \quad \text{if } \forall(\phi_i\ a_i) \in \sum,\ \omega \in [\phi_i]$$
$$= 1 - \max\{ a_i : (\phi_i\ a_i) \in \sum \text{ and } \omega \notin [\phi_i] \} \quad \text{otherwise.}$$

The possibility distribution $\pi_{\sum}$ is not necessarily normalized, however $\pi_{\sum}$ is normalized iff Σ is consistent. Moreover, it can be verified that:

$$Inc(\Sigma) = 1 - \max_{\omega} \pi_{\sum}(\omega).$$

Lastly, syntactic possibiistic inference is very efficient with a complexity close to the one of classical logic.

6.2. Syntactic counterpart of conditioning

Let Σ be a possibilistic knowledge base, and π_Σ its associated possibility distribution (using the above definition). This subsection provides a syntactic counterpart of conditioning π_Σ with some observation A. This consists in constructing from a possibilistic base Σ and the new information A, a new possibilistic base Σ' such that:

$$\forall\omega,\ \pi_{\Sigma'}(\omega) = \pi_\Sigma(\omega|A).$$

This is done in a very simple way: add the input A to the knowledge base with highest possible priority (i.e., 1); compute the level of inconsistency $x = Inc(\Sigma\cup\{(A, 1)\})$ of the resulting possibly inconsistent knowledge base; drop all formulas with priority less than or equal to this level of inconsistency. This guarantees that the remaining beliefs are consistent with A. More formally, Σ' is defined as follows:

$$\Sigma' = \{(\phi_i,\ a_i) : (\phi_i,\ a_i)\in\Sigma \text{ and } a_i > x\} \cup \{(A, 1)\}.$$

6.3. Syntactic counterpart of filtering

Let Σ_t be a knowledge base associated to π_t (using the above definition). Given a prediction function f and a new observation A, the new possibility distribution is computed in two steps:

i) apply conditioning of π_t to A. Let π' be the result of this step;
ii) compute π_{t+1} using the function f in the following way:

$$\pi_{t+1}(\omega) = \max_{\omega':\omega=f(\omega')} \pi'(\omega').$$

Now we are interested in constructing Σ_{t+1} such that :

$$\pi_{\Sigma_{t+1}}(\omega) = \pi_{t+1}(\omega)\ .$$

The first step (i) above can be easily done using the above subsection. Let Σ' be the syntactic result of this step, with $\alpha_n = 1 > \alpha_{n-1} > \ldots > \alpha_1$ (with let $\alpha_0=0$) as the weights used in Σ' and we denote by S_i be the set of classical formulas having the weight equal to α_i. We now describe π'

in terms of classes corresponding to the same certainty level, that we denote C_i, and defined as follows:

$$C_0 = [S_1 \cup \ldots \cup S_n]$$
$$C_i = [S_{i+1} \cup \ldots \cup S_n] - [S_i \cup \ldots \cup S_n], \qquad \text{for } i=1,n-1,$$
$$C_n = \{\text{countermodels of } S_n\},$$

where $[\phi]$ denotes classical models of ϕ.

We can easily check that the C_i's encodes exactly the possibility distribution associated to Σ', namely we have:

$$\pi'(\omega) = 1 - \alpha_i \text{ iff } \qquad \omega \in C_i .$$

We are taking advantage of the compatibility of the extension principle with the level cutting of Σ'. Let us describe similarly π_{t+1} using the classes C_i's, and the function f. This can be done very simply in the following way:

$$\{\omega : \pi_{t+1}(\omega)=1-\alpha_i \} = f(C_i) - \cup_{j=0,i-1} f(C_j), \qquad \text{for } i = 0, n-1$$

and

$$\{\omega : \pi_{t+1}(\omega)=0\} = \Omega - \cup_{j=0,n-1} f(C_j),$$

where $f(C_i) = \{f(\omega) : \omega \in C_i\}$.

Given this representation, the knowledge base Σ_{t+1} associated to π_{t+1} can be easily defined. Let ξ_i be a classical formula whose models is the set $f(C_i) - \cup_{j=0,i-1} f(C_j)$, and let ξ_n be a classical formula whose models is the set $\Omega - \cup_{j=0,n-1} f(C_j)$. Then Σ_{t+1} is defined as follows:

$$\Sigma_{t+1} = \{(\xi_n, 1)\} \cup \{(\xi_i, \alpha_i) : i=1,n-1\}.$$

Applying the steps in a different order, a syntactic counterpart to updating can be easily obtained in a similar way. See also (Dubois and Prade, 1996).

7. Conclusion

This paper has presented a very preliminary investigation of the idea of filtering in the qualitative setting of possibility theory and possibilistic logic setting. In spite of some similarities, filtering and updating have been contrasted. Their respective roles for situation assessment and for acknowledging the dynamics of the world are still to be better analyzed.

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