Coherentist Contraction

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1. Introduction

Coherence is one of the epistemic ideals most often referred to in the belief revision literature, not least since the treatment of coherence in Peter Gärdenfors's seminal book *Knowledge in Flux*. The purpose of the present paper is to investigate what can be meant by performing a contraction coherently.

A minimal requirement is that contraction should be *coherence preserving*, i.e. if the original belief state was coherent, then the contraction outcome should also be coherent. Consider the following simple example: I presently believe that a colleague in Buenes Aires has read the letter that I wrote to him two weaks ago (α). Furthermore, I believe that I posted the said letter after I wrote it (β). There are various ways in which my belief state may be contracted, i.e. so changed that my set of beliefs is reduced to a subset of the original set of beliefs. It holds for all coherent belief states that I can arrive at through contraction that if α is retained, then so is β. Obviously, there are ways in which I can be brought to coherently believing α but not β, but these involve the acquisition of some new belief (such as the belief that a friend brought a copy of the letter on a trip to Argentina, etc.).

The example shows that not all subsets of my current belief set can be arrived at through coherentist contraction, and this simple insight will be our starting-point for formal developments. The properties of the coherent subsets of the belief set are discussed in Section 2 and, based on that, a series of postulates for coherentist contraction are introduced in Section 3. In Section 4, surprising connections between coherentist contraction and "foundationalist" models of belief change are reported, and in Section 5 some philosophical implications of these connections are discussed. The proofs are left out in this version of the paper.

2. Coherent subsets of the belief set

Our basic apparatus will include a language *L* that is closed under truth-functional operations and a consequence operator Cn with the usual properties. (Hansson 1999, p. 26) Furthermore, we will assume that the original belief state is represented by a consistent and logically closed set **K** of sentences. We will assign to **K** a set $\mathfrak{C}$ of logically closed subsets of **K**, that represent the coherent subsets of **K**. The outcome of coherentist contraction should be an element of $\mathfrak{C}$.

As was indicated above, our main concern is coherence *preservation*. Therefore, we can assume that **K** is coherent, or in other words:

$\mathbf{K} \in \mathfrak{C}$ (*coherent origin*)

Following tradition in belief revision theory, we may assume that only logically true sentences are immune against contraction. Then $\mathfrak{C}$ must satisfy the following postulate:

If $\alpha \notin \mathrm{Cn}(\varnothing)$, then there is some X such that $\alpha \notin X \in \mathfrak{C}$.

Equivalently:

$\cap\mathfrak{C} \subseteq \mathrm{Cn}(\varnothing)$

A stronger requirement is that Cn(ø) be among the coherent sets. To have no contingent beliefs at all seems to be a coherent, although unproductive option. We then have:

$\mathrm{Cn}(\varnothing) \in \mathfrak{C}$ (*coherent void*)

The term “coherent void” was used by Olsson (1997) for a related property.

Since the original belief set **K** is coherent, we should expect its coherent parts to be coherent in combination. Let *X* be the belief set consisting of my beliefs about human evolution and *Y* that consisting of my beliefs about religion. *X* and *Y* are both subsets of my belief set **K**. Assuming that **K**, *X*, and *Y* are all coherent, we can reasonably expect the combination of *X* and *Y* to be coherent.

Since we use belief sets to represent coherent belief states, the coherent “combination” of *X* and *Y* referred to here must be represented by a belief set, namely $\mathrm{Cn}(X \cup Y)$. The following notation is convenient:

Definition D1: $X \hat{\cup} \mathrm{Y} = \mathrm{Cn}(X \cup Y)$

We can use it to express the property discussed above as follows:

If $X, Y \in \mathfrak{C}$, then $X \hat{\cup} Y \in \mathfrak{C}$. ($\hat{\cup}$-*closure*)

The corresponding property for intersection immediately suggest itself:

If $X, Y \in \mathfrak{C}$, then $X \cap Y \in \mathfrak{C}$. ($\cap$-*closure*)

Contrary to $X \cup Y$, $X \cap Y$ is logically closed if both *X* and *Y* are so. Therefore, no operation analogous to $\hat{\cup}$ needs to be introduced.

Due to the logical closure of belief sets, $\cap$-closure is more plausible than it might first seem to be. The following may at first seem to be a counterexample: In my present (coherent) state of belief, I believe that my American friend Andy has Swedish ancestors (α). I also believe both that Andy’s mother has Swedish ancestors (β) and that his father has Swedish ancestors (δ). Hence, α, β, and δ are all elements of **K**. It is reasonable to assume that there is some coherent subset *X* of **K** in which α and β are both elements but not δ, and also some other coherent subset

Y in which α and δ are both elements but not β. Then $X \cap Y$ contains α but neither β nor δ, which seems incoherent. However, due to the logical closure of X and Y, $X \cap Y$ contains $\beta \vee \delta$. It is perfectly coherent to believe in α and $\beta \vee \delta$ but neither in β nor δ.

The properties of $\mathfrak{C}$ that we have now introduced combine into an algebraic structure:

> **Definition D2:** Let **K** be a consistent and logically closed set. $\mathfrak{C}$ is a $\hat{\cup}$–*semi-lattice* for **K** if and only if it is a set of logically closed subsets of **K** such that:
>
> (1) $\mathrm{Cn}(\varnothing) \in \mathfrak{C}$ (*coherent void*),
>
> (2) $\mathbf{K} \in \mathfrak{C}$ (*coherent origin*), and
>
> (3) For all $X, Y \in \mathfrak{C}$, $X \hat{\cup} Y \in \mathfrak{C}$ ($\hat{\cup}$-*closure*).
>
> Furthermore, if it also satisfies
>
> (4) For all $X, Y \in \mathfrak{C}$, $X \cap Y \in \mathfrak{C}$ ($\cap$-*closure*),
>
> then it is a $\hat{\cup}\cap$-lattice for **K**.

Due to the cognitive limitations of actual epistemic agents, we can realistically assume that $\mathfrak{C}$ is finite, i.e. has a finite number of elements. For the same reason, we may assume that each element of $\mathfrak{C}$ has a finite representation.

> **Definition D3:** Any subset $\mathfrak{C}$ of $P(L)$ is
>
> (1) *finite* if and only if it has a finite number or elements.
>
> (2) *finitely representable* if and only if for each $X \in \mathfrak{C}$ there is some finite set X' such that $X = \mathrm{Cn}(X')$.
>
> (3) *finitistic* if and only if it is both finite and finitely representable.

If the language L is finite, then $\mathfrak{C}$ is finitistic. On the other hand, $\mathfrak{C}$ may be finitistic without L being finite.

3. Postulates for coherentist contraction

The purpose of introducing the set $\mathfrak{C}$ of coherent subsets of **K** was that all contraction outcomes should be elements of $\mathfrak{C}$. This requirement amounts to the following postulate for coherentist contraction:

$\mathbf{K}\div A \in \mathfrak{C}$ (*coherence*)

Here, A is a subset of L. In the terminology of Fuhrmann and Hansson (1994), ÷ is a *multiple operation* on **K**; by this is meant that it takes subsets of L, rather than elements of L, as inputs. Given that the elements of $\mathfrak{C}$ are logically closed subsets of **K**, it follows from the coherence postulate that two of the basic Gärdenfors postulates (Alchourrón et al 1985, Gärdenfors 1988) are satisfied, namely inclusion ($\mathbf{K}\div A \subseteq \mathbf{K}$) and closure ($\mathbf{K}\div A = \mathrm{Cn}(\mathbf{K}\div A)$).

Another elementary property of contraction operators is lacking. It remains to ensure that A is absent from $\mathbf{K}\div A$ whenever this is possible. To express this we have use for the following notation:

Definition D4: $X \vdash_{\exists} Y$ if and only if there is some $y \in Y$ such that $X \vdash y$.

The postulate can now be expressed as follows:

If $\bigcap\mathfrak{C} \nvdash_{\exists} A$, then $(\mathbf{K}\div A) \nvdash_{\exists} A$. (*success*)

The antecedent of this postulate is equivalent with: "If there is some $X \in \mathfrak{C}$ such that $X \nvdash_{\exists} A$". Furthermore, if $\mathfrak{C}$ satisfies coherent void, then it is equivalent with "If $\varnothing \nvdash_{\exists} A$".

The outcome of coherentist contraction should be determined by the requirements of coherence. When writing this, I believe that this year's Cristmas Eve is a Friday (α) and that this year's Christmas Day is a Saturday (β). All coherent subsets of my belief set have either both α and β as elements or neither of them. With respect to coherentist contraction, these two beliefs stand or fall together. Therefore, contraction by α and contraction by β should be have a uniform treatment, i.e. $\mathbf{K}\div\{\alpha\} = \mathbf{K}\div\{\beta\}$. This argument can be generalized into the following postulate:

> If it holds for all $X \in \mathfrak{C}$ that $X \vdash_{\exists} A$ if and only if $X \vdash_{\exists} A'$, then $\mathbf{K}\div A = \mathbf{K}\div A'$. (*symmetry*)

Finally, coherentist contraction should be conservative, i.e. it should avoid unnecessary losses of information. The postulates introduced up to now are compatible with the mutilating operation ÷ such that for all A, $\mathbf{K}\div A = \mathrm{Cn}(\varnothing)$. An obvious, but rather strong version of the conservativity criterion consists in requiring that $\mathbf{K}\div A$ be inclusion-maximal among those elements of $\mathfrak{C}$ that do not imply any element of A:

> If $\mathbf{K}\div A \subset X \in \mathfrak{C}$, then $\mathbf{K}\div A \nvdash_{\exists} A$ and $X \vdash_{\exists} A$. (*maximality*)

Maximality is quite demanding. In particular, it does not allow for ties between equally plausible maximal elements of $\mathfrak{C}$. Presently I believe that Annie is Mary's daughter (α) and also that Beata is Mary's daughter (β). Suppose that I find reasons to doubt that Mary has two daughters, and as a result of this I give up my belief in $\alpha\&\beta$. The resulting new belief set may contain at most one of α and β. Now suppose that my original reasons for believing α were exactly as strong as those for believing β, so that I have no non-arbitrary way of choosing only one of them to be retained after contraction. I will then have to give up both of them. The resulting belief set $\mathbf{K}\div\{\alpha\&\beta\}$ will not satisfy our maximality postulate. More precisely, there will be some X such that $\mathbf{K}\div\{\alpha\&\beta\} \subset X \in \mathfrak{C}$, $X \nvdash \alpha\&\beta$, and $\alpha \in X$, and similarly there will be some Y such that $\mathbf{K}\div\{\alpha\&\beta\} \subset Y \in \mathfrak{C}$, $Y \nvdash \alpha\&\beta$, and $\beta \in Y$. The reason why $\mathbf{K}\div\{\alpha\&\beta\}$ is not replaced by its proper superset X can be explained in terms of the competitor Y and the fact that $X\cup Y \vdash \alpha\&\beta$. The weakened form of the maximality postulate runs as follows:

If $\mathbf{K}\div A \subset X \in \mathbb{C}$, then there is some $Y \in \mathbb{C}$ such that $\mathbf{K}\div A \subseteq Y \nvdash_{\exists} A$ and $X \cup Y \vdash_{\exists} A$. (*weak maximality*)

4. The surprising connection

The most influential model of belief contraction is *partial meet contraction*, that is part of the AGM model (Alchourrón et al 1985). It takes $\mathbf{K}\div\alpha$ to be the intersection of a selection (the "best") of the maximal consistent subsets of **K** that do not imply any element of α. The following series of definitions introduces partial meet contraction in a slightly more general way than the original AGM publication, since it employs sets rather than single sentences as contraction inputs. (For a systematic treatment of this generalization, see Fuhrmann and Hansson 1994.)

Definition D5: (Alchourrón and Makinson 1981) Let *B* and *A* be sets of sentences. The set $B\perp A$ ("*B* remainder *A*") is the set of sets such that $X \in B\perp A$ if and only if:

(1) $X \subseteq B$,

(2) $X \nvdash_{\exists} A$, and

(3) There is no set *X'* such that $X \subset X' \subseteq B$ and $X' \nvdash_{\exists} A$.

Definition D6: (Alchourrón et al 1985) Let **K** be a belief set. A *selection function* for **K** is a function γ such that for all sets *A* of sentences:

(1) If $\mathbf{K}\perp A$ is non-empty, then $\gamma(\mathbf{K}\perp A)$ is a non-empty subset of $\mathbf{K}\perp A$, and

(2) If $\mathbf{K}\perp A$ is empty, then $\gamma(\mathbf{K}\perp A) = \{\mathbf{K}\}$.

Definition D7: (Alchourrón et al 1985) Let **K** be a belief set and γ a selection function for **K**. The *partial meet contraction* on **K** that is generated by γ is the operation $\sim_\gamma$ such that for all sets *A* of sentences:

$$\mathbf{K}\sim_\gamma A = \cap\gamma(\mathbf{K}\perp A)$$

An operation ÷ on **K** is a partial meet contraction if and only if there is a selection function γ such that for all sets A of sentences: $\mathbf{K}\div A = \mathbf{K}\sim_{\gamma}A$.
$\sim_{\gamma}$ is a *maxichoice contraction* if and only if for all A, $\gamma(\mathbf{K}\perp A)$ has exactly one element.

An alternative approach to belief change is *belief base dynamics* based on partial meet contraction. (Hansson 1994) In this approach, the selection function operates on a belief base B for **K**. (Any set B such that $\mathrm{Cn}(B) = \mathbf{K}$ is a belief base for **K**.) Definitions D6-D7 can be used here as well; we just have to replace the belief set **K** by a belief base (arbitrary set of sentences) B in the definitions. In addition, we can define the (derived) operation on a belief set **K** that is based on a partial meet contraction on some base B for **K**:

Definition D8: (Hansson 1993) An operator ÷ for a belief set **K** is a *base-generated partial meet contraction* if and only if there is a belief base B for **K** and an operator $\sim_{\gamma}$ of partial meet contraction for B such that for all sets A of sentences: $\mathbf{K}\div A = \mathrm{Cn}(B\sim_{\gamma} A)$.

It is commonly assumed that the original AGM framework, which applies partial meet contraction directly to the belief set, represents a coherentist view of belief change whereas the belief base approach corresponds to foundationalist epistemology. (See references in Hansson and Olsson 1999.) Against this background, it is surprising to find that the two algebraic structures for coherent subsets of the belief set that we introduced in Section 2 both correspond exactly to belief base structures:

Definition D9: $B\perp^{\exists} = \{X \mid X \in B\perp A \text{ for some } A\}$

Theorem T1: Let **K** be a consistent and logically closed set. Then the following four conditions on a set $\mathfrak{C}$ are equivalent:
(1) $\mathfrak{C}$ is a finitistic $\hat{\cup}$–semi-lattice on **K**.

(2) $\mathfrak{C} = \{\text{Cn}(X) \mid X \in B\perp^{\exists}\}$ for some finite base B of **K**.

(3) $\mathfrak{C} = \{\text{Cn}(X) \mid (\exists W)(X = \cap\gamma(B\perp W))\}$ for some finite base B of **K** and selection function γ for B.

(4) $\mathfrak{C} = \{\text{Cn}(X) \mid (\exists W)(X = \cap(B\perp W))\}$ for some finite base B of **K**.

Theorem T2: Let **K** be a consistent and logically closed set. Then the following four conditions on a set $\mathfrak{C}$ are equivalent:

(1) $\mathfrak{C}$ is a finitistic $\dot{\cup}\cap$-lattice on **K**.

(2) $\mathfrak{C} = \{\text{Cn}(X) \mid X \in B\perp^{\exists}\}$ for some disjunctively closed finite base B of **K**.

(3) $\mathfrak{C} = \{\text{Cn}(X) \mid (\exists W)(X = \cap\gamma\,(B\perp W))\}$ for some disjunctively closed finite base B of **K** and selection function γ for B.

(4) $\mathfrak{C} = \{\text{Cn}(X) \mid (\exists W)(X = \cap(\,B\perp W))\}$ for some disjunctively closed finite base B of **K**.

A belief base B is disjunctively closed if and only if it holds for all sentences α and β that if $\alpha, \beta \in B$, then $\alpha\vee\beta \in B$.

But there is more to come. Not only $\mathfrak{C}$, but also the contraction operator ÷ can be connected to belief base structures. The following four theorems show that coherentist contraction, as defined in Section 3, coincides with base-generated partial meet contraction.

Theorem T3: Let **K** be a consistent and logically closed subset of L, and let ÷ be a multiple operation on **K**. Then the following two conditions are equivalent:

(1) There is a finitistic $\dot{\cup}$–semi-lattice $\mathfrak{C}$ for **K** such that for all $A \subseteq L$:

- (a) **K**÷$A \in \mathfrak{C}$ (*coherence*)
- (b) If $\cap\mathfrak{C} \nvdash_{\exists} A$, then (**K**÷$A$) $\nvdash_{\exists} A$. (*success*)
- (c) If it holds for all $X \in \mathfrak{C}$ that $X \vdash_{\exists} A$ if and only if $X \vdash_{\exists} A'$, then **K**÷A = **K**÷A'. (*symmetry*)

(d) If $\mathbf{K}\div A \subset X \in \mathfrak{C}$, then $\mathbf{K}\div A \nvdash_{\exists} A$ and $X \vdash_{\exists} A$. (*maximality*)

(2) There is a finite base B for $\mathbf{K}$ and a maxichoice selection function γ for B such that for all $A \subseteq L$:
$\mathbf{K}\div A = \mathrm{Cn}(\cap\gamma(B\perp A))$

Theorem T4: Let $\mathbf{K}$ be a consistent and logically closed subset of L, and let $\div$ be a multiple operation on $\mathbf{K}$. Then the following two conditions are equivalent:

(1) There is a finitistic $\hat{\cup}\cap$-lattice $\mathfrak{C}$ for $\mathbf{K}$ such that for all $A \subseteq L$

(a) $\mathbf{K}\div A \in \mathfrak{C}$ (*coherence*)

(b) If $\cap\mathfrak{C} \nvdash_{\exists} A$, then $(\mathbf{K}\div A) \nvdash_{\exists} A$. (*success*)

(c) If it holds for all $X \in \mathfrak{C}$ that $X \vdash_{\exists} A$ if and only if $X \vdash_{\exists} A'$, then $\mathbf{K}\div A = \mathbf{K}\div A'$. (*symmetry*)

(d) If $\mathbf{K}\div A \subset X \in \mathfrak{C}$, then $\mathbf{K}\div \nvdash_{\exists} A$ and $X \vdash_{\exists} A$. (*maximality*)

(2) There is a disjunctively closed finite base B for $\mathbf{K}$ and a maxichoice selection function γ for B such that for all $A \subseteq L$:
$\mathbf{K}\div A = \mathrm{Cn}(\cap\gamma(B\perp A))$

Theorem T5: Let $\mathbf{K}$ be a consistent and logically closed subset of L, and let $\div$ be a multiple operation on $\mathbf{K}$. Then the following two conditions are equivalent:

(1) There is a finitistic $\hat{\cup}$–semi-lattice $\mathfrak{C}$ for $\mathbf{K}$ such that for all $A \subseteq L$:

(a) $\mathbf{K}\div A \in \mathfrak{C}$ (*coherence*)

(b) If $\cap\mathfrak{C} \nvdash_{\exists} A$, then $(\mathbf{K}\div A) \nvdash_{\exists} A$. (*success*)

(c) If it holds for all $X \in \mathfrak{C}$ that $X \vdash_{\exists} A$ if and only if $X \vdash_{\exists} A'$, then $\mathbf{K}\div A = \mathbf{K}\div A'$. (*symmetry*)

(d) If $\mathbf{K}\div A \subset X \in \mathfrak{C}$, then there is some $Y \in \mathfrak{C}$ such that $\mathbf{K}\div A \subseteq Y \nvdash_{\exists} A$ and $X \cup Y \vdash_{\exists} A$. (*weak maximality*)

(2) There is a finite base B for $\mathbf{K}$ and a selection function γ for B such that for all $A \subseteq L$:
$\mathbf{K}\div A = \mathrm{Cn}(\cap\gamma(B\perp A))$

Theorem T6: Let **K** be a consistent and logically closed subset of L, and let ÷ be a multiple operation on **K**. Then the following two conditions are equivalent:

(1) There is a finitistic $\hat{\cup}\cap$-lattice $\mathfrak{C}$ for **K** such that for all $A \subseteq L$:

(a) **K**÷$A \in \mathfrak{C}$ (*coherence*)

(b) If $\cap\mathfrak{C} \nvdash_{\exists} A$, then $(\mathbf{K}÷A) \nvdash_{\exists} A$. (*success*)

(c) If it holds for all $X \in \mathfrak{C}$ that $X \vdash_{\exists} A$ if and only if $X \vdash_{\exists} A'$, then **K**÷A = **K**÷A'. (*symmetry*)

(d) If $\mathbf{K}÷A \subset X \in \mathfrak{C}$, then there is some $Y \in \mathfrak{C}$ such that $\mathbf{K}÷A \subseteq Y \nvdash_{\exists} A$ and $X \cup Y \vdash_{\exists} A$. (*weak maximality*)

(2) There is a disjunctively closed finite base B for **K** and a selection function γ for B such that for all $A \subseteq L$:

$\mathbf{K}÷A = \mathrm{Cn}(\cap\gamma(B\perp A))$

5. Conclusion

We started out to investigate coherentist contraction. We began by identifying a set of plausible properties for the set $\mathfrak{C}$ of coherent subsets of the original belief set **K**. (Section 2.) We then combined these with a set of plausible postulates for coherentist contraction based on $\mathfrak{C}$. (Section 3) The outcome of this whole exercise turned out to coincide with base-generated partial meet contraction, that is usually seen as the epitome of foundationalism in belief change theory. How can a coherentist construction be equivalent with a foundationalist one? Something seems to be wrong here.

As far as I can see, what we have to modify is the conventional view that coherentist epistemology corresponds to belief set models and foundationalist epistemology to models that make essential use of belief bases. As has been argued in detail elsewhere (Hansson and Olsson 1999), the justificatory relationships that keep a coherent belief system together do not have the structure exhibited by the logical relationships (induced by logical closure) that hold between the sentences in a belief set.

Furthermore, if the coherentist interpretation of the AGM model is taken to mean that the outcome of partial meet contraction, as applied directly to a belief set, is always coherent, then we are forced to accept the conclusion that all logically closed subsets of the original belief set are coherent. The reason for this is that if **K** is a belief set and $\sim_\gamma$ a partial meet contraction on **K**, then for all logically closed subsets **K'** of **K**, there is some *A* such that **K'** = **K**$\sim_\gamma A$. (This follows Observation 3 of Hansson 1995.) This results amounts to a trivialization of coherence that seems difficult to combine with coherentist epistemology.

Belief bases have more expressive power than belief sets. This paper has shown that their expressive power can be used for modelling some elementary properties of coherentist contraction. In order to obtain more fine-tuned models of coherentist belief change, more sophisticated tools may be needed, especially to represent justificatory relationships.

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