

Disbelief Change*

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Abstract

In the paper we address the problem of change of belief state of a rational agent. In our representation of belief state both beliefs and disbeliefs are taken into account. On the belief part, the starting point is the *AGM* theory [1, 5, 6]. On the disbelief part, we build a counterpart of the *AGM* model starting with the rejection consequence Cn' introduced in [2, 4]. Next we investigate changes of pairs of sets of formulae representing belief states.

Keywords: *AGM* theory, rejection consequence.

1 Introduction

Since antiquity, logic and epistemology have traditionally dealt with the 'positive' dimensions of knowledge and reasoning, to the detriment of what one might call the 'negative' dimensions. The focus has been on what is known, believed, conjectured, perceived as true, rather than that which is rejected as false. And logic is usually presented as a theory about inferring true conclusions from true premises; only rarely is it considered as a theory about how to reject new propositions on the basis of old [2, 4, 8, 9, 14, 11, 12, 13, 15, 16, 17]. Even in probabilistic reasoning, the concept of evidence against a hypothesis is usually reduced to the notion of evidence in favour of its formal negation, rather than treated as a separate primitive concept in its own right.

The traditional approach may have a certain appeal when dealing, say, with scientific knowledge encoded by sets of empirical laws. It appears much less

*It is a great pleasure to be able to dedicate this paper to Peter Gärdenfors on the occasion of his 50th birthday. Although as far as we know, Peter's published work on theory change only deals with *belief* revision and related notions, about ten years ago he and the second author discussed briefly the topic of disbelief revision together and looked at potential postulates. Unfortunately, that project was soon abandoned; but we hope the present paper marks a fitting if late successor to it. The authors would like to thank John Cantwell, Sten Lindström, Wlodek Rabinowicz, Tor Sandqvist, and last but not least, to Krister Segerberg for their helpful remarks on earlier versions and presentations of this work. All errors left are our sole responsibility.

convincing when other forms of human knowledge and reasoning are taken into account. In many everyday contexts we act, react and plan on the basis of rejected propositions. We reason from given disbeliefs to further disbeliefs, or, say in deontic contexts, from one set of prohibitions to another. Often this occurs in direct and unmediated fashion, without the conscious intervention of 'positive' beliefs, the formal negations of propositions, and so on.

Given the importance of 'negative' propositional attitudes, like rejection and disbelief, it seems worthwhile to consider the formal reconstruction and analysis of these types of concept. The present paper is a first attempt to consider the problem of revising disbeliefs as a process analogous to but formally independent from the process of belief revision. We approach the subject in two stages. In the first we consider postulates governing the expansion, contraction and revision of disbeliefs, and operations that satisfy these postulates. This analysis may be applied whenever it seems useful or indeed preferable to formalise a given topic, context or problem area directly in terms of a disbelief system. But here, on the whole, we expect results to be rather analogous to those obtained for standard approaches to belief revision, such as *AGM*. The second stage is somewhat more ambitious. Here we consider 'mixed' systems, where beliefs and disbeliefs are each represented explicitly and independently in one framework. Now the revision process may apply to one or other of the two components, and it may apply in such a way that both components are affected, e.g., a simple and consistent expansion of beliefs may nevertheless require a contraction of disbeliefs, if the total system is to remain coherent in the sense that no statements are both believed and disbelieved concurrently.

An excursion into the field of disbelief change will have scant value if, ultimately, the 'negative' dimensions simply collapse, boolean style, into a trivial mirror of their 'positive' counterparts. On the contrary, we hope to be able to express more subtle distinctions of (dis)belief change, precisely by including a non-redundant, disbelief component. There seem to be at least two ways of achieving this objective. One would be to apply a nonclassical underlying logic where the notion of rejection is appropriately handled as a distinct item independent from acceptance. An alternative, followed here, is to take a largely classical approach to the underlying logic, but to relax the relations between belief and disbelief; in particular we do not assume formal connections between the acts of retracting a belief and asserting the corresponding disbelief, nor between expanding the set of beliefs by adding a proposition and expanding the set of disbeliefs by adding the negation of that proposition. Indeed, where beliefs and disbeliefs of an agent obtain as outcomes of an aggregation algorithm, they may be to some extent independent from each other. Such situations can happen when information comes from several (or more) different sources or an agent is a collective one.¹

¹For instance, if an agent is a group of other agents.

Example 1.1 Consider a group of k experts (or sources of information). Let α represent a piece of information. Suppose that m of k experts state that α holds, while n of them state that α does not hold. Nobody may state that α holds and does not hold simultaneously. However, some of the experts may have no opinion of α , i.e., $m + n \leq k$.

The fraction of positive (resp., negative) answers $\frac{m}{k}$ (resp., $\frac{n}{k}$) may be seen as the degree of certainty of the group of experts that α holds (does not hold²). Assume that this degree is 1 (resp., 0) for tautologies (countertautologies); and that logically equivalent formulae are given the same degree. Let $cr(\alpha) \in [0, 1]$ denote the degree of certainty that α holds (or the degree of credibility of α [3]). In our case $cr(\alpha) = \frac{m}{k}$ and $cr(\neg\alpha) = \frac{n}{k}$. Clearly, $cr(\alpha) + cr(\neg\alpha) \leq 1$.

In the next step the group may arrive at an opinion about α in qualitative terms as 'believed' or 'disbelieved'. Suppose two numbers $0 < t_1 \leq t_2 \leq 1$ are considered as threshold values for disbelief and belief, respectively.³ More precisely, a piece of information α is believed if $t_2 \leq cr(\alpha)$ and disbelieved if $cr(\alpha) < t_1$. In the remaining case where $t_1 \leq cr(\alpha) < t_2$, α is neither believed nor disbelieved.

Proposition 1.2 If $1 < t_1 + t_2$ and α is believed, then $\neg\alpha$ is disbelieved.

Nevertheless, from the fact that α is disbelieved one cannot say whether $\neg\alpha$ is believed or not. Moreover, if $t_1 + t_2 \leq 1$ (e.g., $t_1 = 0.3$ and $t_2 = 0.6$), it can be the case that α is believed (e.g., $cr(\alpha) = 0.62$) and $\neg\alpha$ is not disbelieved (e.g., $cr(\neg\alpha) = 0.35$).

In this paper, our underlying logic will be based on the first author's work on rejection systems [2, 4] and we shall take an *AGM*-style approach to the revision process [1, 5, 6]. Belief states of rational agents will be represented by pairs of sets (X, Y) of formulae closed under the classical consequence operation Cn and the rejection consequence operation Cn' [2, 4], respectively. All considered changes of belief state will be 'small', i.e. by a single formula. We postulate the result of change of a belief state to be a coherent belief state whenever possible. We hope in future work to be able to explore other ways of analysing (dis)belief change and compare them with the present one.

The paper is organized as follows. Section 2 contains preliminaries. Section 3 is devoted to the consequence operations Cn and Cn' . In Section 4 we address the problem of consistency and coherence. Apart from the definitions and properties of consistency and coherence, we give definitions of some mappings used later on to build our model of change of belief state, recall the concept of consolidation [7] renamed here to b-consolidation, introduce the notions of d-consolidation and a disjoining operation, and investigate their properties. In Section 5 we briefly review the *AGM* theory. As a novelty we define the notion

²For simplicity, this may be identified with the case that $\neg\alpha$ holds.

³Clearly, threshold values may be subject to change and depend, e.g., on the considered problem.

of b-incorporation. In Section 6 we propose a model of disbelief change. In Section 7 we complete our model to be able to speak about changes of belief state. In particular, we define the notions of B- and D-expansions, incorporations, revisions, and revisions, relate them to one another, and investigate their properties. Section 8 contains brief final remarks.

2 Preliminaries

First, we introduce the notation used in the paper. Given a set X , let $\wp(X)$ (resp., $\text{card}(X)$) denote the power set (cardinality) of X . For any sets X and Y , $X - Y$ denotes their set-theoretical difference. Later on the same symbol will be used in other contexts as well. As usual, ' $x, y \in X$ ' is a shortening for ' $x \in X$ and $y \in X$ '. Given a mapping $f : X \mapsto Y$ and a set $Z \subseteq X$, $f|_Z$ denotes the limitation of f to Z , viz., $f \cap (Z \times Y)$. Instead of $X \times X$ (resp., the composition of f with f) we shall often write X^2 (resp., f^2) if convenient. The parentheses $(,)$ will be omitted if no confusion results. The set of all natural numbers is denoted by \mathcal{N} as usual.

A relation \preceq on X is a partial ordering and (X, \preceq) is a partially ordered set if \preceq is reflexive (i.e., $(\forall x \in X) x \preceq x$), antisymmetric $((\forall x, y \in X)((x \preceq y \wedge y \preceq x) \rightarrow x = y))$, and transitive $((\forall x, y, z \in X)((x \preceq y \wedge y \preceq z) \rightarrow x \preceq z))$. If \preceq is also connected $((\forall x, y \in X)(x \preceq y \vee y \preceq x))$, then it is called a total ordering and (X, \preceq) is a totally ordered set. An element $x \in X$ is called \preceq -*maximal* (resp., \preceq -*minimal*)⁴ in (X, \preceq) if $(\forall y \in X)(x \preceq y \rightarrow x = y)$ (resp., $(\forall y \in X)(y \preceq x \rightarrow x = y)$).

Given a partially ordered set (X, \subseteq) , define a *product inclusion* \sqsubseteq on the set $(\wp(X))^2$ as follows:

$$(X_1, Y_1) \sqsubseteq (X_2, Y_2) \text{ iff } X_1 \subseteq X_2 \wedge Y_1 \subseteq Y_2 \quad (1)$$

(X_1, Y_1) is called a *subpair* of (X_2, Y_2) . It is easy to show that \sqsubseteq is a partial ordering on $(\wp(X))^2$.

In a natural way we generalise the set-theoretical union \cup (resp., intersection \cap) to an operation \sqcup (resp., \sqcap) to deal with pairs of sets:

$$\begin{aligned} (X, Y) \sqcup (Z, T) &= (X \cup Z, Y \cup T) \\ (X, Y) \sqcap (Z, T) &= (X \cap Z, Y \cap T) \end{aligned} \quad (2)$$

Let π_i denote the *projection* function onto the i -th variable ($i = 1, 2$). Thus for any sets X_i and elements $x_i \in X_i$, $\pi_i : X_1 \times X_2 \mapsto X_i$ and $\pi_i(x_1, x_2) = x_i$. For any $Z \subseteq X_1 \times X_2$, $\pi_i^{-1}(Z)$ is the image of Z given by π_i :

$$\pi_i^{-1}(Z) = \{x \in X_i \mid (\exists (x_1, x_2) \in Z) \pi_i(x_1, x_2) = x\} \quad (3)$$

⁴Usually the prefix ' \preceq ' is omitted if it is known from the context.

Clearly, $\pi_i^{-1}(X_1 \times X_2) = X_i$.

In our approach we use the language of the classical propositional logic (*PC* for short) to formalise and investigate the notions of beliefs, disbeliefs, and change of belief state. Thus information, beliefs, and disbeliefs are uniformly represented by formulae of the *PC* language. Propositional variables (resp., formulae) are denoted by p, q (the lowercase Greek letters α, β, γ) with sub/superscripts if needed. The propositional connectives of our language (and the meta-language if no confusion arises) are denoted by $\wedge, \vee, \rightarrow, \leftrightarrow$, and \neg , and have the usual meaning. We use 0 (resp., 1) to abbreviate $p \wedge \neg p$ (resp., $p \vee \neg p$). *FOR* denotes the set of all formulae formed according to the usual rules. For any set of formulae X , we define $\neg X$ as follows:

$$\neg X = \{\neg \alpha \mid \alpha \in X\} \quad (4)$$

Notice that $\neg \emptyset = \emptyset$.

By a *selector* we mean a mapping $\xi : \wp^2(\text{FOR}) \mapsto \wp^2(\text{FOR})$ such that for any family of sets of formulae \mathcal{X} , the following conditions are satisfied:

$$\begin{aligned} \xi(\mathcal{X}) &\subseteq \mathcal{X} \\ \xi(\mathcal{X}) = \emptyset &\text{ iff } \mathcal{X} = \emptyset \end{aligned} \quad (5)$$

The notion of selector may be generalised to the case of pairs of families of sets of formulae as follows. A *p-selector* ('p' for 'product' or 'pair') is a mapping $\tau : (\wp^2(\text{FOR}))^2 \mapsto (\wp^2(\text{FOR}))^2$ such that for any families of sets of formulae \mathcal{X}_i ($i = 1, 2$), the following conditions are satisfied:

$$\begin{aligned} \tau(\mathcal{X}_1, \mathcal{X}_2) &\subseteq (\mathcal{X}_1, \mathcal{X}_2) \\ \pi_i \tau(\mathcal{X}_1, \mathcal{X}_2) = \emptyset &\text{ iff } \mathcal{X}_i = \emptyset \end{aligned} \quad (6)$$

3 Operations Cn and Cn'

In this section we recall general properties of consequence operators. Apart from the well-known classical propositional consequence operator Cn , we also consider a propositional rejection consequence Cn' [2, 4].

The notation $(Y, \alpha) \in r$ is used to indicate that Y is a finite set of *premises* and α is the *conclusion* of an inference rule r . For any set of inference rules R and any set of formulae X , a *syntactical consequence* operator CN may be defined as follows [10]:

$$\begin{aligned} CN^0(R, X) &= X \\ CN^{n+1}(R, X) &= CN^n(R, X) \cup \{\alpha \mid (\exists r \in R)(\exists Y \subseteq CN^n(R, X))(Y, \alpha) \in r\} \\ CN(R, X) &= \bigcup_{n \in \mathbb{N}} CN^n(R, X) \end{aligned} \quad (7)$$

Proposition 3.1 *For any set of inference rules R and sets of formulae X, Y , we have that:*

$$\begin{aligned}
X &\subseteq CN(R, X) \text{ (reflexivity)} \\
\text{If } X \subseteq Y, &\text{ then } CN(R, X) \subseteq CN(R, Y). \text{ (monotonicity)} \\
CN(R, CN(R, X)) &\subseteq CN(R, X) \text{ (idempotence)} \\
CN(R, X) &= \bigcup \{CN(R, Y) \mid Y \subseteq X \wedge \text{card}(Y) < \aleph_0\} \\
&\text{(compactness)}
\end{aligned}$$

□

Thus, CN satisfies the Tarski postulates for a consequence operator.

Proposition 3.2 *Let R be a set of inference rules and \mathcal{X} a family of sets of formulae. (1) If (\mathcal{X}, \subseteq) is totally ordered, then we have that:*

$$CN(R, \bigcup \mathcal{X}) = \bigcup \{CN(R, X) \mid X \in \mathcal{X}\}$$

(2) *If for each $X \in \mathcal{X}$, $CN(R, X) = X$, then we have that:*

$$CN(R, \bigcap \mathcal{X}) = \bigcap \mathcal{X}$$

□

We shall consider two inference rules: the modus ponens rule $MP : \frac{\alpha, \alpha \rightarrow \beta}{\beta}$ and the *rejection rule* Rej introduced in [2, 4] to formalise the notion of rejection consequence.⁵ Let $\alpha - \beta = \neg(\alpha \rightarrow \beta)$. Rej is defined schematically as $Rej : \frac{\beta, \alpha - \beta}{\alpha}$ and understood as follows: *If β and $\alpha - \beta$ are rejected, then α is rejected as well.*

Two consequence operators will be considered: the classical one denoted by Cn and the *rejection consequence operator* Cn' [2, 4]. Let X be a set of formulae, Ax denote a sound and complete set of axioms of PC , and $Sb(Ax)$ the set of all substitution instances of Ax . Operators Cn and Cn' are defined by the following equations:

$$Cn(X) = CN(\{MP\}, Sb(Ax) \cup X) \quad (8)$$

$$Cn'(X) = CN(\{Rej\}, \neg Cn(\emptyset) \cup X) \quad (9)$$

$Cn(\emptyset)$ is the set of all PC tautologies. It turns out that $Cn'(\emptyset) = \{\alpha \mid \neg\alpha \in Cn(\emptyset)\}$, i.e., it is the set of all PC countertautologies. Henceforth C will denote Cn or Cn' . We say that a set of formulae X is C -closed (or is a C -theory) in case $C(X) = X$. A pair of sets of formulae (X, Y) is *closed* if X is Cn -closed and Y is Cn' -closed. When speaking of beliefs and disbeliefs, Cn -closed (resp.,

⁵For other formalisations of rejection inference see, e.g., [9, 14, 15, 16, 17].

Cn' -closed) sets of formulae will be referred to as *belief (disbelief) sets*, and closed pairs of sets of formulae will be called *belief states* as well.

Below we recall some useful properties of Cn and Cn' .⁶

Proposition 3.3 *For any set of formulae X and formulae α , β , and γ , the following conditions hold:*

$$\begin{aligned} \text{If } \alpha \leftrightarrow \beta \in Cn(\emptyset), \text{ then } \alpha \in Cn'(X) \text{ iff } \beta \in Cn'(X). \\ \text{If } \alpha \in Cn'(X) \text{ or } \beta \in Cn'(X), \text{ then } \alpha \wedge \beta \in Cn'(X). \\ \text{If } \alpha \in Cn'(X) \text{ or } \neg\beta \in Cn'(X), \text{ then } \alpha - \beta \in Cn'(X). \\ \alpha \vee \beta \in Cn'(X) \quad \text{iff} \quad \alpha, \beta \in Cn'(X) \\ \alpha - \beta \in Cn'(X) \quad \text{iff} \quad \neg\beta - \neg\alpha \in Cn'(X) \end{aligned}$$

□

Operators Cn and Cn' are closely related.

Proposition 3.4 *For any set of formulae X , we have that:*

$$\begin{aligned} C(X) &= C(\neg\neg X) \\ \neg Cn(X) &\subseteq Cn'(\neg X) \\ \neg Cn'(X) &\subseteq Cn(\neg X) \end{aligned}$$

□

Hence for any formula α , $\alpha \in Cn'(X)$ iff $\neg\alpha \in Cn(\neg X)$.

Where X is finite, $\bigwedge X$ (resp., $\bigvee X$) denotes a conjunction (disjunction) of all formulae of X .

Theorem 3.5 *For any sets of formulae X , Y , a finite set of formulae Z , and formulae α , β , we have that:*

$$\begin{aligned} \alpha \in Cn'(X \cup \{\beta\}) \quad \text{iff} \quad \alpha - \beta \in Cn'(X) \\ \text{If } \alpha \leftrightarrow \beta \in Cn(\emptyset), \text{ then } C(X \cup \{\alpha\}) = C(X \cup \{\beta\}). \\ Cn(\{\bigwedge Z\}) &= Cn(Z) \\ Cn(\{\bigvee Z\}) &= \bigcap \{Cn(\{\alpha\}) \mid \alpha \in Z\} \\ Cn'(\{\bigwedge Z\}) &= \bigcap \{Cn'(\{\alpha\}) \mid \alpha \in Z\} \\ Cn'(\{\bigvee Z\}) &= Cn'(Z) \end{aligned}$$

□

⁶For more details on properties of Cn' a reader is referred to [4].

4 Consistency and coherence

Let us recall that C denotes Cn or Cn' . A set of formulae X is C -consistent if $C(X) \neq FOR$; otherwise it is C -inconsistent. Let $Cons(X)$ (resp., $Cons'(X)$) iff X is Cn -consistent (Cn' -consistent). A pair of sets of formulae (X, Y) is consistent in case $Cons(X)$ and $Cons'(Y)$; otherwise it is inconsistent.

Some results regarding consistency are presented below.

Theorem 4.1 *For any sets of formulae X, Y and a formula α , the following conditions hold:*

$$\begin{aligned} C(X) = FOR & \quad \text{iff} \quad (\exists \alpha)(\alpha, \neg \alpha \in C(X)) \\ Cons(X) & \quad \text{iff} \quad Cons'(\neg X) \\ C(X \cup \{\neg \alpha\}) \neq FOR & \quad \text{iff} \quad \alpha \notin C(X) \\ Cn(X) \cap Cn'(Y) = \emptyset & \quad \text{iff} \quad Cons(X \cup \neg Y) \end{aligned}$$

□

Notice that $Cons(X)$ (resp., $Cons'(X)$) iff $0 \notin Cn(X)$ (resp., $1 \notin Cn'(X)$).

We slightly change the definition of the mapping \perp given in [1] to allow $X \perp \alpha = \{X\}$ if $\alpha \in Cn(\emptyset)$. We also attach the subscript 'b' for 'belief'. Thus, $\perp_b: \wp(FOR) \times FOR \mapsto \wp^2(FOR)$ is a mapping defined as follows:

$$X \perp_b \alpha = \begin{cases} \{Y \mid Y \subseteq X \text{ is maximal s.t. } \alpha \notin Cn(Y)\} & \text{if } \alpha \notin Cn(\emptyset) \\ \{X\} & \text{otherwise} \end{cases} \quad (10)$$

Clearly, if $\alpha \notin Cn(X)$, then $X \perp_b \alpha = \{X\}$. According to the definition, $X \perp_b 0$ is the family of all \subseteq -maximal Cn -consistent subsets of X . A mapping $\perp_d: \wp(FOR) \times FOR \mapsto \wp^2(FOR)$ is defined in a similar way:

$$X \perp_d \alpha = \begin{cases} \{Y \mid Y \subseteq X \text{ is maximal s.t. } \alpha \notin Cn'(Y)\} & \text{if } \alpha \notin Cn'(\emptyset) \\ \{X\} & \text{otherwise} \end{cases} \quad (11)$$

Note that if $\alpha \notin Cn'(X)$, then $X \perp_d \alpha = \{X\}$. Moreover, $X \perp_d 1$ is the family of all \subseteq -maximal Cn' -consistent subsets of X .

Some basic properties of \perp_b and \perp_d are given below.

Proposition 4.2 *For any set of formulae X and formulae α, β , we have that:*

$$\begin{aligned} \text{If } Cn(X) = X, \quad \text{then} \quad & (\forall Y \in X \perp_b \alpha) Cn(Y) = Y. \\ \text{If } Cn'(X) = X, \quad \text{then} \quad & (\forall Y \in X \perp_d \alpha) Cn'(Y) = Y. \\ \text{If } \alpha \leftrightarrow \beta \in Cn(\emptyset), \quad \text{then} \quad & X \perp_b \alpha = X \perp_b \beta \text{ and } X \perp_d \alpha = X \perp_d \beta. \\ \text{If } Cn(X) = X, \quad \text{then} \quad & (X \perp_b \alpha) \cap (X \perp_b \beta) \subseteq X \perp_b (\alpha \wedge \beta) \\ & \subseteq (X \perp_b \alpha) \cup (X \perp_b \beta). \\ \text{If } Cn'(X) = X, \quad \text{then} \quad & (X \perp_d \alpha) \cap (X \perp_d \beta) \subseteq X \perp_d (\alpha \vee \beta) \end{aligned}$$

$$\begin{array}{lll}
& \subseteq & (X \perp_d \alpha) \cup (X \perp_d \beta). \\
X \perp_b 0 \neq \emptyset & \text{and} & X \perp_d 1 \neq \emptyset \\
X \perp_b 0 = \{X\} & \text{iff} & \text{Cons}(X) \\
X \perp_d 1 = \{X\} & \text{iff} & \text{Cons}'(X)
\end{array}$$

□

The notion of *consolidation* (renamed to *b-consolidation* here) was proposed by Hansson [7]. Roughly speaking, it is an operation of making a given set of formulae *Cn*-consistent. Given a selector ξ (cf. (5)), a *partial-meet b-consolidation* generated by ξ is a mapping $\odot_b^\xi : \wp(FOR) \mapsto \wp(FOR)$ defined as follows:

$$\odot_b^\xi(X) = \bigcap \xi(X \perp_b 0) \quad (12)$$

By Proposition 4.2, $\odot_b^\xi(X)$ is a *Cn*-consistent belief set whenever X is a belief set.

Similarly, a *partial-meet d-consolidation* generated by ξ is a mapping $\odot_d^\xi : \wp(FOR) \mapsto \wp(FOR)$ defined as follows:

$$\odot_d^\xi(X) = \bigcap \xi(X \perp_d 1) \quad (13)$$

Let us note that where X is a disbelief set, $\odot_d^\xi(X)$ is a *Cn'*-consistent disbelief set.

We can distinguish two limiting cases. If ξ is the identity function, *id*, \odot_b^{id} and \odot_d^{id} are called the *full-meet b-* and *d-consolidations*, respectively. On the other hand, if ξ selects a single set, the generated partial-meet *b-* and *d-consolidations* are called *maxi-choice b-* and *d-consolidations*, respectively.

In the case of pairs of sets of formulae, the notion of consistency can be too weak. For instance, a belief state (X, Y) may be consistent according to the formal definition although $X \cap Y \neq \emptyset$. This means that there is a formula both believed and disbelieved, contrary to the intuition.

Example 4.3 Let $X = Cn(\{p, p \rightarrow q\})$ and $Y = Cn'(\{q\})$. In this case, $q \in X \cap Y$.

Therefore we introduce a notion of coherence which is more appropriate for our purposes. A pair of sets of formulae (X, Y) is called *coherent* if $Cn(X) \cap Cn'(Y) = \emptyset$; otherwise (X, Y) is *incoherent*. By Theorem 4.1, (X, Y) is coherent iff $\text{Cons}(X \cup \neg Y)$. Henceforth we shall denote the set of all coherent pairs of sets of formulae by *Coh*:

$$\text{Coh} = \{(X, Y) \subseteq FOR^2 \mid \text{Cons}(X \cup \neg Y)\} \quad (14)$$

Mappings $\perp_b^p, \perp_d^p : (\wp(FOR))^2 \times FOR \mapsto \wp((\wp(FOR))^2)$ defined below are generalisations of the mappings \perp_b and \perp_d , respectively, to the case of pairs of

sets of formulae.

$$(X, Y) \perp_b^p \alpha = \begin{cases} \{(Z, T) \sqsubseteq (X, Y) \mid (Z, T) \text{ is maximal s.t.} \\ (Z, T) \in Coh \wedge \alpha \notin Cn(Z)\} & \text{if } \alpha \notin Cn(\emptyset) \\ \{(X, Y)\} & \text{otherwise} \end{cases} \quad (15)$$

$$(X, Y) \perp_d^p \alpha = \begin{cases} \{(Z, T) \sqsubseteq (X, Y) \mid (Z, T) \text{ is maximal s.t.} \\ (Z, T) \in Coh \wedge \alpha \notin Cn'(T)\} & \text{if } \alpha \notin Cn'(\emptyset) \\ \{(X, Y)\} & \text{otherwise} \end{cases} \quad (16)$$

Thus, $(X, Y) \perp_b^p \alpha$ (resp., $(X, Y) \perp_d^p \alpha$) is the set of all \sqsubseteq -maximal coherent subpairs (Z, T) of (X, Y) such that $\alpha \notin Cn(Z)$ (resp., $\alpha \notin Cn'(T)$) if α is not a tautology (countertautology); and it is $\{(X, Y)\}$ otherwise. In particular, $(X, Y) \perp_b^p 0$ or, equivalently, $(X, Y) \perp_d^p 1$ is the set of all \sqsubseteq -maximal coherent subpairs of (X, Y) .

Proposition 4.4 *For any sets of formulae X, Y , a formula α , and $i \in \{b, d\}$, we have that:*

$$\begin{aligned} & \text{If } (X, Y) \text{ is closed, then } (Z, T) \text{ is closed for each } (Z, T) \in (X, Y) \perp_i^p \alpha. \\ & \text{If } \alpha \leftrightarrow \beta \in Cn(\emptyset), \text{ then } (X, Y) \perp_i^p \alpha = (X, Y) \perp_i^p \beta. \\ & (X, Y) \perp_b^p 0 \neq \emptyset \\ & (X, Y) \perp_b^p 0 = \{(X, Y)\} \quad \text{iff} \quad (X, Y) \in Coh \\ & \text{If } (X, Y) \in Coh, \text{ then } (X, Y) \perp_b^p \alpha = (X \perp_b \alpha) \times \{Y\} \\ & \text{and } (X, Y) \perp_d^p \alpha = \{X\} \times (Y \perp_d \alpha). \end{aligned}$$

□

Given a (possibly incoherent) belief state, we may want to transform it into a coherent one. For this purpose we introduce the notion of disjoining operation playing a similar role as consolidation. A *disjoining operation* is a mapping $\odot : (\wp(FOR))^2 \mapsto (\wp(FOR))^2$ such that for any sets of formulae X, Y , the following postulates are satisfied:

- ($\odot 1$) $\odot(X, Y) \sqsubseteq (X, Y)$
- ($\odot 2$) $\odot(X, Y) \sqsubseteq \odot \odot(X, Y)$
- ($\odot 3$) $(X, Y) \sqsubseteq \odot(X, Y)$ iff $(X, Y) \in Coh$.
- ($\odot 4$) If (X, Y) is closed, then $\odot(X, Y)$ is closed.

Proposition 4.5 *For any disjoining operation \odot and sets of formulae X, Y , $\odot(X, Y) \in Coh$.* □

Hence, $\odot(X, Y)$ is a coherent belief state whenever (X, Y) is a belief state.

A *partial-meet disjoining* operation $\odot^\tau : (\wp(FOR))^2 \mapsto (\wp(FOR))^2$, generated by a p-selector τ (cf. (6)), is defined as follows:

$$\odot^\tau(X, Y) = (\bigcap \pi_1^\rightarrow \tau((X, Y) \perp_b^p 0), \bigcap \pi_2^\rightarrow \tau((X, Y) \perp_b^p 0)) \quad (17)$$

Intuitively, τ selects some \sqsubseteq -maximal coherent subpairs of (X, Y) which form a set Z . Next, we take the pair (X_1, Y_1) , where $X_1 = \bigcap \pi_1^\rightarrow(Z)$ and $Y_1 = \bigcap \pi_2^\rightarrow(Z)$, as $\odot^\tau(X, Y)$. Notice that $(X_1, Y_1) \in Coh$.

Two particular cases are worth distinguishing. If τ is the identity function id , \odot^{id} is referred to as the *full-meet disjoining* operation. On the other hand, if τ always selects a single pair of sets of formulae, \odot^τ is called a *maxi-choice disjoining* operation.

Proposition 4.6 *Any partial-meet disjoining operation is a disjoining operation, i.e., it satisfies the postulates $(\odot 1)$ – $(\odot 4)$.* \square

If the result of disjoining is to be \sqsubseteq -maximal, a supplementary postulate may be formulated as follows:

$$(\odot 5) \quad (\forall (U, W) \sqsubseteq (X, Y))(((U, W) \in Coh \wedge \odot(X, Y) \sqsubseteq (U, W)) \rightarrow (U, W) = \odot(X, Y)) \quad (18)$$

One can see that maxi-choice disjoining operations satisfy $(\odot 5)$ but the full-meet disjoining operation does not.

A mapping $\| : (\wp(FOR))^2 \times FOR \mapsto \wp((\wp(FOR))^2)$, being in some sense 'orthogonal' to \perp_b^p and \perp_b^p , is defined as follows:

$$(X, Y) \| \alpha = \begin{cases} \{(Z, T) \sqsubseteq (X, Y) \mid (Z, T) \text{ is maximal s.t.} \\ (Z \cup \{\alpha\}, T) \in Coh\} & \text{if } \alpha \notin Cn'(\emptyset) \\ \{(X, Y)\} & \text{otherwise} \end{cases} \quad (19)$$

In words, $(X, Y) \| \alpha$ is the set of all \sqsubseteq -maximal subpairs (Z, T) of (X, Y) such that $(Z \cup \{\alpha\}, T)$ is coherent if α is not a countertautology; and it is $\{(X, Y)\}$ otherwise.

Proposition 4.7 *For any sets of formulae X, Y and a formula α , we have that:*

$$\begin{aligned} & \text{If } (X, Y) \text{ is closed, then } (Z, T) \text{ is closed for each } (Z, T) \in (X, Y) \| \alpha. \\ & \text{If } \alpha \leftrightarrow \beta \in Cn(\emptyset), \text{ then } (X, Y) \| \alpha = (X, Y) \| \beta. \\ & (X, Y) \| 1 \neq \emptyset \\ & (X, Y) \| 1 = \{(X, Y)\} \text{ iff } (X, Y) \in Coh \end{aligned}$$

\square

In general, $(X, Y) \parallel \alpha$ cannot be defined as $(X, Y) \perp_b^p \neg\alpha$, $(X, Y) \perp_d^p \alpha$ or $(X \perp_b \neg\alpha) \times (Y \perp_d \alpha)$ as shown in the example below.

Example 4.8 Let $X = \{p\}$, $Y = \{\neg(p \rightarrow q)\}$, and $\alpha = \neg q$. One can easily see that (1) $\text{Cons}(X \cup \neg Y)$, (2) $\text{Cons}(X \cup \{\alpha\})$, and (3) $\text{Cons}(\{\alpha\} \cup \neg Y)$. On the other hand, (4) $\neg\text{Cons}(X \cup \{\alpha\} \cup \neg Y)$ and (5) $\alpha \notin \text{Cn}'(\emptyset)$. $X \perp_b \neg\alpha = \{X\}$ by (2), $Y \perp_d \alpha = \{Y\}$ by (3), $(X, Y) \perp_b^p \neg\alpha = \{(X, Y)\} = (X, Y) \perp_d^p \alpha$ by (1) and Proposition 4.4, and $(X, Y) \notin (X, Y) \parallel \alpha$ by (4) and (5). In summary, we have that:

$$((X, Y) \parallel \alpha) \cap (((X, Y) \perp_b^p \neg\alpha) \cup ((X, Y) \perp_d^p \alpha) \cup ((X \perp_b \neg\alpha) \times (Y \perp_d \alpha))) = \emptyset$$

5 Belief change

In this section we recall the *AGM* operations of belief change (i.e., expansion, contraction, and revision) [1]. For the sake of uniformity, these operations will be presented as b-expansion ($+_b$), b-contraction ($-_b$), and b-revision ($*_b$), respectively. We also define a general form of belief incorporation, b-incorporation (\circ_b) following the idea of Hansson [7].

Given a belief set X and a formula α , representing a piece of information, two major situations of change of belief state can be distinguished. First we believe α and try to incorporate it into X . Secondly, we stop believing α and try to remove it from X . The main postulate for belief change is that the result has to be a *Cn*-consistent belief set. The second form of belief change may be formally described as a b-contraction of X by α . The first one may be formalised as a b-incorporation being a composition of a b-consolidation defined in the preceding section and the b-expansion. If we also require the result of incorporating α into X to contain α (a rule known as the *priority-to-novelty principle*), the belief change may be realised by a b-incorporation of a particular kind called b-revision.

5.1 b-expansion and b-incorporation

By a *b-expansion* we mean a mapping $+_b : \wp(\text{FOR}) \times \text{FOR} \mapsto \wp(\text{FOR})$ defined as follows:

$$X +_b \alpha = \text{Cn}(X \cup \{\alpha\}) \quad (20)$$

Notice that the results of b-expansion are always belief sets. Unfortunately, $X +_b \alpha$ may or may not be *Cn*-consistent. Therefore not every case of incorporation of a belief into a belief set may be realised by the b-expansion.

As observed by Hansson [7], incorporation of α into X proceeds in two steps if no priority is given to α just because it is a new belief. First, X is expanded with α . Next, the result is consolidated appropriately. A mapping realising such a change may be defined as follows. Given a selector ξ (cf. (5))

and the generated partial-meet b-consolidation \odot_b^ξ (cf. (12)), a *partial-meet b-incorporation* generated by ξ is a mapping $\circ_b^\xi : \wp(FOR) \times FOR \mapsto \wp(FOR)$ defined as follows:

$$X \circ_b^\xi \alpha = \odot_b^\xi(X +_b \alpha) \quad (21)$$

The result of b-incorporating α into X , written $X \circ_b^\xi \alpha$, is a *Cn*-consistent belief set whenever X is a belief set. It may or may not contain α . For pairs (X, α) such that $\neg\alpha \notin Cn(X)$ or, in other words, $Cns(X \cup \{\alpha\})$, b-incorporation takes the form of b-expansion. As in the case of b-consolidation we can distinguish two limiting cases. If \odot_b^ξ is the full-meet (resp., a maxi-choice) b-consolidation, the corresponding mapping \circ_b^ξ is referred to as the *full-meet* (a *maxi-choice*) *b-incorporation*.

5.2 b-contraction

Consider a belief set X and a formula α . According to the definition of a belief set, all tautologies are believed forever. Thus if α is a tautology, we cannot successfully remove it from X . In the remaining case α is to be removed from X (possibly with some other formulae) if we stop believing it. As postulated before, the obtained belief set should be *Cn*-consistent. Moreover, it should not contain α . Such a change of X can be formalised as a b-contraction of X by α . In detail, a *b-contraction* is a mapping $-_b : \wp(FOR) \times FOR \mapsto \wp(FOR)$ such that for any set of formulae X and formulae α, β , the following *rationality postulates* are satisfied [1]:

- (-_b1) If $Cn(X) = X$, then $Cn(X -_b \alpha) = X -_b \alpha$. (*closure*)
- (-_b2) $X -_b \alpha \subseteq X$ (*inclusion*)
- (-_b3) If $\alpha \notin Cn(X)$, then $X -_b \alpha = X$. (*vacuity*)
- (-_b4) If $\alpha \notin Cn(\emptyset)$, then $\alpha \notin Cn(X -_b \alpha)$. (*success*)
- (-_b5) If $\alpha \leftrightarrow \beta \in Cn(\emptyset)$, then $X -_b \alpha = X -_b \beta$. (*preservation*)
- (-_b6) If $Cn(X) = X$, then $X \subseteq (X -_b \alpha) +_b \alpha$. (*recovery*)

$X -_b \alpha$ denotes the result of b-contracting X by α .

In our approach partial-meet contractions [1] are presented as partial-meet b-contractions. A *partial-meet b-contraction* generated by a selector ξ is a mapping $-_b^\xi : \wp(FOR) \times FOR \mapsto \wp(FOR)$ defined as follows:

$$X -_b^\xi \alpha = \bigcap \xi(X \perp_b \alpha) \quad (22)$$

In general, selector ξ chooses some maximal $Y \subseteq X$ such that α is not *Cn*-derived from Y if α is not a tautology; and ξ yields $\{X\}$ otherwise. In the next step, we take the intersection of all selected sets as $X -_b^\xi \alpha$. It turns out that partial-meet b-contractions are b-contractions as well [1].

Theorem 5.1 *Any partial-meet b-contraction is a b-contraction, i.e., it satisfies the postulates $(-b1)$ – $(-b6)$.* \square

Along the standard lines, two cases of partial-meet b-contractions can be distinguished. If ξ is the identity function id , then $-_b^{id}$ is called the *full-meet b-contraction*. In this case $X -_b^{id} \alpha = \bigcap (X \perp_b \alpha)$. If ξ always selects a single set, then the generated partial-meet b-contraction $-_b^\xi$ is called a *maxi-choice b-contraction*.

In [1] two supplementary postulates for contraction are proposed. They are satisfied by some classes of partial-meet contractions only. In our notation the postulates take the following form. Let X be a Cn -closed set of formulae and α, β be formulae.

$$(-b7) \quad (X -_b \alpha) \cap (X -_b \beta) \subseteq X -_b (\alpha \wedge \beta)$$

$$(-b8) \quad X -_b (\alpha \wedge \beta) \subseteq X -_b \alpha \text{ whenever } \alpha \notin X -_b (\alpha \wedge \beta)$$

5.3 b-revision

In this section we consider the case of incorporation of a believed formula α into a belief set X under an additional condition⁷ that α has to be in the obtained belief set. This particular kind of b-incorporation is called b-revision (or, simply, *revision* [1, 5, 6]). Formally, by a *b-revision* we mean a mapping $*_b : \wp(FOR) \times FOR \mapsto \wp(FOR)$ such that for any set of formulae X and formulae α, β , the following *rationality postulates* [5] are satisfied:

$$(*b1) \quad Cn(X *_b \alpha) = X *_b \alpha$$

$$(*b2) \quad \alpha \in X *_b \alpha$$

$$(*b3) \quad X *_b \alpha \subseteq X +_b \alpha$$

$$(*b4) \quad \text{If } \neg\alpha \notin Cn(X), \text{ then } X +_b \alpha \subseteq X *_b \alpha.$$

$$(*b5) \quad \text{If } \alpha \leftrightarrow \beta \in Cn(\emptyset), \text{ then } X *_b \alpha = X *_b \beta.$$

$$(*b6) \quad X *_b \alpha = FOR \text{ iff } \neg\alpha \in Cn(\emptyset)$$

The result of b-revising X by α is written as $X *_b \alpha$. b-revisions by α may produce Cn -inconsistent belief sets only if α is a countertautology. In the remaining case, $X *_b \alpha$ is a Cn -consistent belief set containing α . Let us notice that b-revisions take the form of b-expansion for pairs (X, α) such that $\neg\alpha \notin Cn(X)$ or equivalently $Cons(X \cup \{\alpha\})$.

The notions of b-contraction and b-revision are interrelated. Let ξ be a selector and $-_b^\xi$ the generated partial-meet b-contraction. By a *partial-meet*

⁷The already mentioned priority-to-novelty principle.

b-revision generated by ξ we mean a mapping $*_b^\xi : \wp(FOR) \times FOR \mapsto \wp(FOR)$ defined as follows:

$$X *_b^\xi \alpha = (X -_b^\xi \neg \alpha) +_b \alpha \text{ (the Levi identity)} \quad (23)$$

Where $-_b^\xi$ is the full-meet (resp., a maxi-choice) b-contraction, the corresponding $*_b^\xi$ is called the *full-meet* (a *maxi-choice*) *b-revision* generated by ξ .

Proposition 5.2 *Mapping $*_b^\xi$ defined above is a b-revision, i.e., it satisfies the postulates $(*_b1)$ – $(*_b6)$. \square*

On the other hand, given a partial-meet b-revision $*_b^\xi$, the mapping $-_b$ defined below is a b-contraction.

$$X -_b \alpha = X \cap (X *_b^\xi \neg \alpha) \text{ (the Harper identity)} \quad (24)$$

Like in the case of b-contraction, two additional postulates for b-revision are proposed in the *AGM* theory. Let X be any set of formulae and α be a formula.

$$(*_b7) \quad X *_b (\alpha \wedge \beta) \subseteq (X *_b \alpha) +_b \beta$$

$$(*_b8) \quad \text{If } \neg \beta \notin X *_b \alpha, \text{ then } (X *_b \alpha) +_b \beta \subseteq X *_b (\alpha \wedge \beta).$$

6 Disbelief change

In this section we define and investigate counterparts of the *AGM* operations and b-incorporation for the case of disbelief change, viz., d-expansion ($+_d$), d-contraction ($-_d$), d-revision ($*_d$), and d-incorporation (\circ_d).

Consider a disbelief set X and a formula α representing a piece of information. As before, two general cases of disbelief change are distinguished. First we disbelieve α and try to incorporate it into X . In the second case we stop disbelieving α what results in trying to remove it from X . As previously, the main postulate says that the result have to be a *Cn'*-consistent disbelief set. Changes of the second kind can be formalised as d-contractions of X by α . Changes of the former one can be realised by d-incorporations, i.e. compositions of d-consolidation and d-expansion. If we additionally require that α is a member of the obtained disbelief set (the priority-to-novelty principle), the changes can be realised by particular d-incorporations called d-revisions.

6.1 d-expansion and d-incorporation

By a *d-expansion* we mean a mapping $+_d : \wp(FOR) \times FOR \mapsto \wp(FOR)$ defined as follows:

$$X +_d \alpha = Cn'(X \cup \{\alpha\}) \quad (25)$$

Let us notice that the result of d-expanding X with α , $X +_d \alpha$, is a disbelief set. It may or may not be Cn' -consistent. Therefore incorporation of α into a disbelief set X , where the only requirement is that the result should be a Cn' -consistent disbelief set, proceeds in two steps: (1) d-expansion of X with α and (2) d-consolidation of the resulting set. Given a selector ξ (cf. (5)) and the generated partial-meet d-consolidation \odot_d^ξ (cf. (13)), a *partial-meet d-incorporation* generated by ξ is a mapping $\circ_d^\xi : \wp(FOR) \times FOR \mapsto \wp(FOR)$ defined as follows:

$$X \circ_d^\xi \alpha = \odot_d^\xi(X +_d \alpha) \quad (26)$$

If X is a disbelief set, the result of d-incorporating α into X , written $X \circ_d^\xi \alpha$, is a Cn' -consistent disbelief set. As in the case of b-incorporation, it may or may not contain α . For pairs (X, α) such that $\neg\alpha \notin Cn'(X)$ or, in other words, $Cons'(X \cup \{\alpha\})$, d-incorporation takes the form of d-expansion. Along the standard lines, two cases of d-incorporation are distinguished. If \odot_d^ξ is the full-meet (resp., a maxi-choice) d-consolidation, the corresponding mapping \circ_d^ξ is called the *full-meet* (a *maxi-choice*) *d-incorporation*.

6.2 d-contraction

Consider a disbelief set X and a formula α . Since all counterautologies are always disbelieved by the definition of a disbelief set, we cannot effectively remove α from X if α is a counterautology. If we stop disbelieving a non-counterautology α , we try to remove it from X , possibly with some other formulae. The result should be a Cn' -consistent disbelief set not containing α . Such a change of X may be realised by a d-contraction of X by α . Formally, a *d-contraction* is a mapping $-_d : \wp(FOR) \times FOR \mapsto \wp(FOR)$ such that for any set of formulae X and formulae α, β , the following postulates are satisfied:

- (-d1) If $Cn'(X) = X$, then $Cn'(X -_d \alpha) = X -_d \alpha$.
- (-d2) $X -_d \alpha \subseteq X$
- (-d3) If $\alpha \notin Cn'(X)$, then $X -_d \alpha = X$.
- (-d4) If $\alpha \notin Cn'(\emptyset)$, then $\alpha \notin Cn'(X -_d \alpha)$.
- (-d5) If $\alpha \leftrightarrow \beta \in Cn(\emptyset)$, then $X -_d \alpha = X -_d \beta$.
- (-d6) If $Cn'(X) = X$, then $X \subseteq (X -_d \alpha) +_d \alpha$.

The result of d-contracting X by α is denoted by $X -_d \alpha$. Like in the *AGM* framework, the postulates for d-contraction may be given special names, viz., *closure* for (-d1), *inclusion* for (-d2), *vacuity* for (-d3), *success* for (-d4), *preservation* for (-d5), and *recovery* for (-d6).

A *partial-meet d-contraction* generated by a selector ξ is a mapping $-_d^\xi : \wp(FOR) \times FOR \mapsto \wp(FOR)$ defined as follows:

$$X -_d^\xi \alpha = \bigcap \xi(X \perp_d \alpha) \quad (27)$$

In a general case ξ selects some maximal $Y \subseteq X$ such that $\alpha \notin Cn'(Y)$ if α is not a countertautology; and ξ yields $\{X\}$ otherwise. Next we take the intersection of all the selected sets as $X -_d^\xi \alpha$.

Theorem 6.1 *Any partial-meet d-contraction is a d-contraction, i.e., it satisfies the postulates $(-d1)$ – $(-d6)$.* \square

Like in the AGM theory we can distinguish two limiting cases: the full-meet d-contraction and the maxi-choice d-contractions. The partial-meet d-contraction generated by id , $-_d^{id}$, is called the *full-meet d-contraction*. In this case $X -_d^{id} \alpha = \bigcap (X \perp_d \alpha)$. On the other hand, if ξ always selects a single set, $-_d^\xi$ is called a *maxi-choice d-contraction*.

We can formulate two additional postulates for d-contraction, $(-d7)$ and $(-d8)$, which correspond to the postulates $(-b7)$ and $(-b8)$, respectively. They are satisfied by some classes of d-contractions only. Let X be any disbelief set and α, β be formulae.

$$(-d7) \quad (X -_d \alpha) \cap (X -_d \beta) \subseteq X -_d (\alpha \vee \beta)$$

$$(-d8) \quad X -_d (\alpha \vee \beta) \subseteq X -_d \alpha \text{ whenever } \alpha \notin X -_d (\alpha \vee \beta)$$

6.3 d-revision

Now we consider the case of incorporation of a disbelieved formula α into a disbelief set X under an additional requirement that α is in the resulting disbelief set (the priority-to-novelty principle). This kind of d-incorporation is called d-revision. In detail, a *d-revision* is a mapping $*_d : \wp(FOR) \times FOR \mapsto \wp(FOR)$ such that for any set of formulae X and formulae α, β , the following postulates are satisfied:

$$(*_d1) \quad Cn'(X *_d \alpha) = X *_d \alpha$$

$$(*_d2) \quad \alpha \in X *_d \alpha$$

$$(*_d3) \quad X *_d \alpha \subseteq X +_d \alpha$$

$$(*_d4) \quad \text{If } \neg\alpha \notin Cn'(X), \text{ then } X +_d \alpha \subseteq X *_d \alpha.$$

$$(*_d5) \quad \text{If } \alpha \leftrightarrow \beta \in Cn(\emptyset), \text{ then } X *_d \alpha = X *_d \beta.$$

$$(*_d6) \quad X *_d \alpha = FOR \text{ iff } \alpha \in Cn(\emptyset).$$

The result of d-revising X by α is written as $X *_d \alpha$. Clearly, for pairs (X, α) such that $\neg\alpha \notin Cn'(X)$ (i.e., $Cons'(X \cup \{\alpha\})$) d-revisions coincide with the d-expansion. The notion of partial-meet d-revision is defined similarly to that of partial-meet b-revision. Given a selector ξ and the generated partial-meet d-contraction $-_d^\xi$, by a *partial-meet d-revision* generated by ξ we mean a mapping $*_d^\xi : \wp(FOR) \times FOR \mapsto \wp(FOR)$ defined as follows:

$$X *_d^\xi \alpha = (X -_d^\xi \neg\alpha) +_d \alpha \quad (28)$$

Where $-_d^\xi$ is the full-meet (resp., a maxi-choice) d-contraction, the corresponding mapping $*_d^\xi$ is called the *full-meet* (a *maxi-choice*) *d-revision*.

Proposition 6.2 *Mapping $*_d^\xi$ defined above is a d-revision, i.e., it satisfies the postulates $(*_d1)$ – $(*_d6)$.* \square

On the other hand, any partial-meet d-revision determines a corresponding d-contraction. Let us define a mapping $-_d : \wp(FOR) \times FOR \mapsto \wp(FOR)$ as follows:

$$X -_d \alpha = X \cap (X *_d^\xi \neg\alpha) \quad (29)$$

Proposition 6.3 *Mapping $-_d$ defined above is a d-contraction, i.e., it satisfies the postulates $(-_d1)$ – $(-_d6)$.* \square

Like in the case of b-revision, we can formulate supplementary postulates satisfied by some classes of d-revisions. Let X be any set of formulae and α be a formula.

$$(*_d7) \quad X *_d (\alpha \vee \beta) \subseteq (X *_d \alpha) +_d \beta$$

$$(*_d8) \quad \text{If } \neg\beta \notin X *_d \alpha, \text{ then } (X *_d \alpha) +_d \beta \subseteq X *_d (\alpha \vee \beta).$$

7 Change of belief states

Let us recall that by belief states we mean closed pairs of sets of formulae, i.e., a pair (X, Y) is a belief state if X is Cn -closed (X is a belief set) and Y is Cn' -closed (Y is a disbelief set). A belief state (X, Y) is consistent just in case X is Cn -consistent and Y is Cn' -consistent. Clearly, consistency does not guarantee disjointness of X, Y . In other words, it can be the case that a belief state (X, Y) is consistent and nevertheless there is a formula α believed and disbelieved concurrently (cf. Example 4.3). In our opinion, such a situation is possible but abnormal. In this paper we are interested in the case of coherent belief states where belief and disbelief sets are disjoint. According to the definition, (X, Y) is coherent iff $Cn(X) \cap Cn'(Y) = \emptyset$. The last equation is equivalent to $X \cap Y = \emptyset$. Thus, we postulate the result of doxastic change of belief state to be a coherent belief state whenever possible.

As previously, we consider small changes, i.e. by a single formula representing a piece of information. We can distinguish two major types of change of belief state: incorporating and removing a formula. Given a belief state (X, Y) and a formula α , we may want to incorporate α into X (resp., Y) since we believe (disbelieve) it. If we stop believing (disbelieving) α , we shall try to remove it from our belief (disbelief) set X (resp., Y). We do not require that believing α should imply disbelieving $\neg\alpha$, or vice versa. As shown in Example 1.1, the notions of belief and disbelief may be quite independent from each other in the case of collective (dis)beliefs or where information comes from several different sources. In the definitions below we use the prefixes 'B' and 'D' for 'belief' and 'disbelief', respectively.

7.1 B-, D-expansions and B-, D-incorporations

B- and *D-expansions* are mappings $\oplus_b, \oplus_d : (\wp(FOR))^2 \times FOR \mapsto (\wp(FOR))^2$, respectively, defined as follows:

$$(X, Y) \oplus_b \alpha = (X +_b \alpha, Y) \text{ and } (X, Y) \oplus_d \alpha = (X, Y +_d \alpha) \quad (30)$$

Note that if (X, Y) is a belief state, the result of B-expanding (X, Y) with α , $(X, Y) \oplus_b \alpha$, is a belief state as well; and similarly for D-expansion. Clearly, $(X, Y) \oplus_b \alpha$ and $(X, Y) \oplus_d \alpha$ may or may not be coherent.

If coherence of the resulting belief state is the only requirement to be fulfilled, incorporation of a formula into a belief state can be realised by B- and D-incorporations which are compositions of disjoining operations with the B- and D-expansions, respectively. Given a disjoining operation \odot , *B-* and *D-incorporations* are mappings $\bullet_b, \bullet_d : (\wp(FOR))^2 \times FOR \mapsto (\wp(FOR))^2$, respectively, defined as follows:

$$(X, Y) \bullet_b \alpha = \odot((X, Y) \oplus_b \alpha) \text{ and } (X, Y) \bullet_d \alpha = \odot((X, Y) \oplus_d \alpha) \quad (31)$$

The result of B-incorporating (resp., D-incorporating) α into a belief state (X, Y) , written $(X, Y) \bullet_b \alpha$ (resp., $(X, Y) \bullet_d \alpha$), is a coherent belief state. For pairs $((X, Y), \alpha)$ such that $(X \cup \{\alpha\}, Y) \in Coh$ (resp., $(X, Y \cup \{\alpha\}) \in Coh$), B-incorporations (D-incorporations) coincide with the B-expansion (D-expansion).

Given a p-selector τ (cf. (6)), a *partial-meet B-incorporation* (resp., *D-incorporation*) generated by τ is a mapping \bullet_b^τ (resp., \bullet_d^τ) defined by (31), where $\odot = \odot^\tau$. The mapping \bullet_b^{id} (resp., \bullet_d^{id}), generated by the identity function *id*, is called the *full-meet B-incorporation* (*D-incorporation*). Where \odot^τ is a maxi-choice disjoining operation, the corresponding mappings \bullet_b^τ and \bullet_d^τ are referred to as maxi-choice B- and D-incorporations, respectively.

7.2 B- and D-contractions

Consider a belief state (X, Y) and a formula α . If α is a tautology (resp., countertautology), there is no way to remove it from X (resp., Y) to obtain a

belief state (Z, T) such that $\alpha \notin Z$ (resp., $\alpha \notin T$) by the definition of a belief (disbelief) set. In the remaining cases, if α is not longer believed (disbelieved), it is to be removed from X (resp., Y), possibly together with some other formulae, in such a way that α cannot be derived from the obtained belief (disbelief) set. Such a change of (X, Y) can be formalised as a B-contraction (D-contraction) of (X, Y) by α . More precisely, *B-* and *D-*contractions are mappings $\ominus_b, \ominus_d : (\wp(FOR))^2 \times FOR \mapsto (\wp(FOR))^2$, respectively, such that for any sets of formulae X, Y and formulae α, β , the following postulates are satisfied:

- (\ominus_b1) If (X, Y) is a belief state, then $(X, Y) \ominus_b \alpha$ is a belief state.
- (\ominus_b2) $(X, Y) \ominus_b \alpha \sqsubseteq (X, Y)$
- (\ominus_b3) If $\alpha \notin Cn(X)$ and $(X, Y) \in Coh$, then $(X, Y) \ominus_b \alpha = (X, Y)$.
- (\ominus_b4) If $\alpha \notin Cn(\emptyset)$, then $\alpha \notin Cn(\pi_1((X, Y) \ominus_b \alpha))$.
- (\ominus_b5) If $\alpha \leftrightarrow \beta \in Cn(\emptyset)$, then $(X, Y) \ominus_b \alpha = (X, Y) \ominus_b \beta$.
- (\ominus_b6) If $Cn(X) = X$ and $(X, Y) \in Coh$, then $X \subseteq \pi_1(((X, Y) \ominus_b \alpha) \oplus_b \alpha)$.
- (\ominus_d1) If (X, Y) is a belief state, then $(X, Y) \ominus_d \alpha$ is a belief state.
- (\ominus_d2) $(X, Y) \ominus_d \alpha \sqsubseteq (X, Y)$
- (\ominus_d3) If $\alpha \notin Cn'(Y)$ and $(X, Y) \in Coh$, then $(X, Y) \ominus_d \alpha = (X, Y)$.
- (\ominus_d4) If $\alpha \notin Cn'(\emptyset)$, then $\alpha \notin Cn'(\pi_2((X, Y) \ominus_d \alpha))$.
- (\ominus_d5) If $\alpha \leftrightarrow \beta \in Cn(\emptyset)$, then $(X, Y) \ominus_d \alpha = (X, Y) \ominus_d \beta$.
- (\ominus_d6) If $Cn'(Y) = Y$ and $(X, Y) \in Coh$, then $Y \subseteq \pi_2(((X, Y) \ominus_d \alpha) \oplus_d \alpha)$.

$(X, Y) \ominus_b \alpha$ and $(X, Y) \ominus_d \alpha$ denote the results of B- and D-contracting (X, Y) by α , respectively.

A simple form of B-contraction called a weak B-contraction can be easily defined by means of b-contraction; and similarly for D-contraction. Given a b-contraction $-_b$ and a d-contraction $-_d$, *weak B- and D-contractions* are mappings $\ominus_{bw}, \ominus_{dw} : (\wp(FOR))^2 \times FOR \mapsto (\wp(FOR))^2$, respectively, defined as follows:

$$(X, Y) \ominus_{bw} \alpha = (X -_b \alpha, Y) \text{ and } (X, Y) \ominus_{dw} \alpha = (X, Y -_d \alpha) \quad (32)$$

Note that if (X, Y) is a belief state, then both $(X, Y) \ominus_{bw} \alpha$ and $(X, Y) \ominus_{dw} \alpha$ are belief states. Similarly, if (X, Y) is coherent, then both $(X, Y) \ominus_{bw} \alpha$ and $(X, Y) \ominus_{dw} \alpha$ are coherent.

Theorem 7.1 *Any weak B-contraction (D-contraction) defined by (32) is a B-contraction (D-contraction), i.e., it satisfies the postulates (\ominus_b1) – (\ominus_b6) (resp., (\ominus_d1) – (\ominus_d6)).* \square

Where $-_b$ (resp., $-_d$) is a partial-meet b-contraction (d-contraction), i.e., $-_b = -_b^\xi$ (resp., $-_d = -_d^\xi$) for some selector ξ , the corresponding weak B-contraction \ominus_{bw}^ξ (D-contraction \ominus_{dw}^ξ) is referred to as a *weak partial-meet B-contraction* (D-contraction) generated by ξ .

When applied to incoherent belief states, weak B- and D-contractions may give incoherent results. Therefore these kinds of contractions are useful as long as coherent belief states are considered. Now we define stronger versions of the above mappings which will always produce coherent pairs of sets of formulae. *Partial-meet B- and D-contractions* generated by a p-selector τ are mappings $\ominus_b^\tau, \ominus_d^\tau : (\wp(FOR))^2 \times FOR \mapsto (\wp(FOR))^2$, respectively, defined as follows:

$$(X, Y) \ominus_b^\tau \alpha = (\bigcap \pi_1^\tau \tau((X, Y) \perp_b^p \alpha), \bigcap \pi_2^\tau \tau((X, Y) \perp_b^p \alpha)) \quad (33)$$

$$(X, Y) \ominus_d^\tau \alpha = (\bigcap \pi_1^\tau \tau((X, Y) \perp_d^p \alpha), \bigcap \pi_2^\tau \tau((X, Y) \perp_d^p \alpha)) \quad (34)$$

Thus in the case of partial-meet B-contraction, τ selects some maximal coherent $(X_1, Y_1) \sqsubseteq (X, Y)$ such that $\alpha \notin Cn(X_1)$. In the next step, we take the families Z_i ($i = 1, 2$) of the i -th elements of the chosen pairs, respectively. That is, $Z_i = \pi_i^\tau \tau((X, Y) \perp_b^p \alpha)$. Finally, we take $(\bigcap Z_1, \bigcap Z_2)$ as the result of B-contracting (X, Y) by α , viz., $(X, Y) \ominus_b^\tau \alpha$. In the case of D-contraction we proceed similarly.

Partial-meet and weak partial-meet B- and D-contractions are closely related for coherent pairs of sets of formulae.

Theorem 7.2 *Let i be b or d in the case of B- and D-contractions, respectively. (1) For every weak partial-meet B-contraction \ominus_{bw}^ξ (resp., D-contraction \ominus_{dw}^ξ), there is a partial-meet B-contraction \ominus_b^τ (D-contraction \ominus_d^τ) such that*

$$\ominus_{i|Coh \times FOR}^\tau = \ominus_{i|Coh \times FOR}^\xi$$

and (2) for every partial-meet B-contraction \ominus_b^τ (resp., D-contraction \ominus_d^τ), there is a weak partial-meet B-contraction \ominus_{bw}^ξ (D-contraction \ominus_{dw}^ξ) such that the above equation holds. \square

To prove the above theorem we apply Proposition 4.4.

Theorem 7.3 *Any partial-meet B-contraction (resp., D-contraction) is a B-contraction (D-contraction), i.e., it satisfies the postulates $(\ominus_b1) - (\ominus_b6)$ (resp., $(\ominus_d1) - (\ominus_d6)$). \square*

Yet stronger forms of B- and D-contractions obtain if we replace ' $\perp_b^p \alpha$ ' and ' $\perp_d^p \alpha$ ' by ' $\parallel \neg \alpha$ ' and ' $\parallel \alpha$ ' in (33) and (34), respectively. *Strong partial-meet B- and D-contractions* generated by τ are mappings $\ominus_{bs}^\tau, \ominus_{ds}^\tau : (\wp(FOR))^2 \times FOR \mapsto (\wp(FOR))^2$, respectively, defined as follows:

$$(X, Y) \ominus_{bs}^\tau \alpha = (\bigcap \pi_1^\tau \tau((X, Y) \parallel \neg \alpha), \bigcap \pi_2^\tau \tau((X, Y) \parallel \neg \alpha)) \quad (35)$$

$$(X, Y) \ominus_{ds}^\tau \alpha = (\bigcap \pi_1^\tau \tau((X, Y) \parallel \alpha), \bigcap \pi_2^\tau \tau((X, Y) \parallel \alpha)) \quad (36)$$

In the case of strong partial-meet B-contraction (resp., D-contraction), τ selects some maximal $(X_1, Y_1) \sqsubseteq (X, Y)$ such that $(X_1 \cup \{\neg\alpha\}, Y_1) \in Coh$ (resp., $(X_1, Y_1 \cup \{\neg\alpha\}) \in Coh$). Next we proceed as in the case of partial-meet B- and D-contractions, respectively.

Theorem 7.4 *Any strong partial-meet B-contraction (resp., D-contraction) is a B-contraction (D-contraction), i.e., it satisfies the postulates (\ominus_b1) – (\ominus_b6) (resp., (\ominus_d1) – (\ominus_d6)).* \square

Let (X, Y) be a belief state and α any formula. Supplementary postulates for B- and D-contractions, respectively, may be formulated as follows:

- (\ominus_b7) $((X, Y) \ominus_b \alpha) \sqcap ((X, Y) \ominus_b \beta) \sqsubseteq (X, Y) \ominus_b (\alpha \wedge \beta)$
- (\ominus_b8) $(X, Y) \ominus_b (\alpha \wedge \beta) \sqsubseteq (X, Y) \ominus_b \alpha$ whenever $\alpha \notin \pi_1((X, Y) \ominus_b (\alpha \wedge \beta))$
- (\ominus_d7) $((X, Y) \ominus_d \alpha) \sqcap ((X, Y) \ominus_d \beta) \sqsubseteq (X, Y) \ominus_d (\alpha \vee \beta)$
- (\ominus_d8) $(X, Y) \ominus_d (\alpha \vee \beta) \sqsubseteq (X, Y) \ominus_d \alpha$ whenever $\alpha \notin \pi_2((X, Y) \ominus_d (\alpha \vee \beta))$

7.3 B- and D-revisions

Given a belief state (X, Y) and a formula α , we again consider the case of incorporation of α into the belief (disbelief) set X (resp., Y) if we believe (disbelieve) α . In this case however, we expect not only the new belief state to be coherent but also α to be a member of the obtained belief (disbelief) set if possible (the priority-to-novelty principle). Such incorporations can be formalised as B- and D-revisions. *B-* and *D-revisions* are mappings $\otimes_b, \otimes_d : (\wp(FOR))^2 \times FOR \mapsto (\wp(FOR))^2$, respectively, such that for any sets of formulae X, Y and formulae α, β , the following postulates are satisfied:

- (\otimes_b1) $(X, Y) \otimes_b \alpha$ is a belief state.
- (\otimes_b2) $\alpha \in \pi_1((X, Y) \otimes_b \alpha)$
- (\otimes_b3) $(X, Y) \otimes_b \alpha \sqsubseteq (X, Y) \oplus_b \alpha$
- (\otimes_b4) If $(X \cup \{\alpha\}, Y) \in Coh$, then $(X, Y) \oplus_b \alpha \sqsubseteq (X, Y) \otimes_b \alpha$.
- (\otimes_b5) If $\alpha \leftrightarrow \beta \in Cn(\emptyset)$, then $(X, Y) \otimes_b \alpha = (X, Y) \otimes_b \beta$.
- (\otimes_b6) $(X, Y) \otimes_b \alpha \notin Coh$ iff $\alpha \in Cn'(\emptyset)$ iff $\pi_1((X, Y) \otimes_b \alpha) = FOR$
- (\otimes_d1) $(X, Y) \otimes_d \alpha$ is a belief state.
- (\otimes_d2) $\alpha \in \pi_2((X, Y) \otimes_d \alpha)$
- (\otimes_d3) $(X, Y) \otimes_d \alpha \sqsubseteq (X, Y) \oplus_d \alpha$
- (\otimes_d4) If $(X, Y \cup \{\alpha\}) \in Coh$, then $(X, Y) \oplus_d \alpha \sqsubseteq (X, Y) \otimes_d \alpha$.

($\otimes_d 5$) If $\alpha \leftrightarrow \beta \in Cn(\emptyset)$, then $(X, Y) \otimes_d \alpha = (X, Y) \otimes_d \beta$.

($\otimes_d 6$) $(X, Y) \otimes_d \alpha \notin Coh$ iff $\alpha \in Cn(\emptyset)$ iff $\pi_2((X, Y) \otimes_d \alpha) = FOR$

The result of B-revising (resp., D-revising) (X, Y) by α is written as $(X, Y) \otimes_b \alpha$ (resp., $(X, Y) \otimes_d \alpha$).

Counterparts of partial-meet b- and d-revisions can be obtained by means of a p-selector and the mapping \parallel given by (19). *Partial-meet B- and D-revisions* generated by a p-selector τ are mappings $\otimes_b^\tau, \otimes_d^\tau : (\wp(FOR))^2 \times FOR \mapsto (\wp(FOR))^2$, respectively, defined as follows:

$$(X, Y) \otimes_b^\tau \alpha = (Cn(\bigcap \pi_1^\tau \tau((X, Y) \parallel \alpha) \cup \{\alpha\}), \bigcap \pi_2^\tau \tau((X, Y) \parallel \alpha)) \quad (37)$$

$$(X, Y) \otimes_d^\tau \alpha = (\bigcap \pi_1^\tau \tau((X, Y) \parallel \neg \alpha), Cn'(\bigcap \pi_2^\tau \tau((X, Y) \parallel \neg \alpha) \cup \{\alpha\})) \quad (38)$$

Thus in the case of partial-meet B-revision, a p-selector τ selects some maximal $(X_1, Y_1) \sqsubseteq (X, Y)$ such that $(X_1 \cup \{\alpha\}, Y_1) \in Coh$. In the next step, we take the families \mathcal{Z}_i ($i = 1, 2$) of the i -th elements of the chosen pairs, i.e., $\mathcal{Z}_i = \pi_i^\tau \tau((X, Y) \parallel \alpha)$. Finally, we take $(Cn(\bigcap \mathcal{Z}_1 \cup \{\alpha\}), \bigcap \mathcal{Z}_2)$ as the result of B-revising (X, Y) by α , viz., $(X, Y) \otimes_b^\tau \alpha$. In the case of partial-meet D-revision we proceed similarly. Observe that $(X, Y) \otimes_b^\tau \alpha$ (resp., $(X, Y) \otimes_d^\tau \alpha$) is coherent if α is not a countertautology (tautology). Moreover, $(X, Y) \otimes_b^\tau \alpha$ and $(X, Y) \otimes_d^\tau \alpha$ are belief states if (X, Y) is a belief state.

Two limiting cases of partial-meet B- and D-revisions are distinguished. \otimes_b^{id} and \otimes_d^{id} are referred to as the *full-meet B-* and *D-revisions*, respectively. If τ always selects exactly one pair of sets, \otimes_b^τ and \otimes_d^τ are called *maxi-choice B-* and *D-revisions*, respectively.

Proposition 7.5 *Any partial-meet B-revision (D-revision) is a B-revision (D-revision), i.e., it satisfies the postulates ($\otimes_b 1$)–($\otimes_b 6$) (resp., ($\otimes_d 1$)–($\otimes_d 6$)).* \square

Like in the AGM theory, we can formulate additional postulates to be satisfied by some B- and D-revisions. Let X, Y be any sets of formulae and α, β be formulae.

($\otimes_b 7$) $(X, Y) \otimes_b (\alpha \wedge \beta) \sqsubseteq ((X, Y) \otimes_b \alpha) \oplus_b \beta$

($\otimes_b 8$) If $((X, Y) \otimes_b \alpha) \oplus_b \beta \in Coh$, then $((X, Y) \otimes_b \alpha) \oplus_b \beta \sqsubseteq (X, Y) \otimes_b (\alpha \wedge \beta)$.

($\otimes_d 7$) $(X, Y) \otimes_d (\alpha \vee \beta) \sqsubseteq ((X, Y) \otimes_d \alpha) \oplus_d \beta$

($\otimes_d 8$) If $((X, Y) \otimes_d \alpha) \oplus_d \beta \in Coh$, then $((X, Y) \otimes_d \alpha) \oplus_d \beta \sqsubseteq (X, Y) \otimes_d (\alpha \vee \beta)$.

Since disjoining operations do not comply with the priority-to-novelty principle in general, B- and D-revisions cannot be defined as compositions of disjoining

operations and b- and d-revisions, respectively. To see this, let \odot be a disjoining operation, $*_b$ a b-revision, $*_d$ a d-revision, and $\bigcirc_b, \bigcirc_d : (\wp(FOR))^2 \times FOR \mapsto (\wp(FOR))^2$ be mappings defined as follows:

$$(X, Y) \bigcirc_b \alpha = \odot(X *_b \alpha, Y) \text{ and } (X, Y) \bigcirc_d \alpha = \odot(X, Y *_d \alpha) \quad (39)$$

As shown below, mappings \bigcirc_b and \bigcirc_d do not satisfy the postulates $(\otimes_b 2)$ and $(\otimes_d 2)$, respectively.

Example 7.6 *Let $X = Cn(\emptyset)$, $Y = Cn'(\{p \vee q\})$, and $\alpha = p$. In this case $X *_b p = X +_b p = Cn(\{p\})$. We have that $X \cap Y = \emptyset$ and $p \vee q \in (X *_b p) \cap Y$. After applying a disjoining operation \odot , $p \vee q \notin \pi_1 \odot (X *_b p, Y)$ or $p \vee q \notin \pi_2 \odot (X *_b p, Y)$. Suppose that the first case holds. Then $p \notin \pi_1 \odot (X *_b p, Y)$, either. That is, $p \notin \pi_1((X, Y) \bigcirc_b p)$ which means that the postulate $(\otimes_b 2)$ is not satisfied by \bigcirc_b . For \bigcirc_d consider $X = Cn(\{p \wedge q\})$, $Y = Cn'(\emptyset)$, and $\alpha = p$.*

However, partial-meet B-revisions (resp., D-revisions) can be defined as compositions of strong partial-meet B-contractions (D-contractions) and B-expansions (D-expansions).

Proposition 7.7 *For any p -selector τ , sets of formulae X, Y , a formula α , and $i \in \{b, d\}$, we have that:*

$$(X, Y) \otimes_i^\tau \alpha = ((X, Y) \ominus_{is}^\tau \neg \alpha) \oplus_i \alpha$$

□

8 Summary

In our approach not only beliefs but also disbeliefs are explicitly taken into account. Indeed, belief states are represented by pairs (X, Y) of sets of formulae of PC , where X is a belief set and Y is a disbelief set. The central problem to us was to identify and formally describe major types of change of such belief states. Thus, we have defined several mappings (viz., B- and D-expansions, incorporations, revisions, and contractions) corresponding to the considered changes. The minimal requirement to be fulfilled by these mappings was the postulate that the result of change of a belief state should be a coherent belief state whenever possible. On the other hand, the problem of minimality of changes, being of a great importance in general, was not of the main interest from the perspective of the present paper. It was not our intention to discuss the question of relevance of the *AGM* theory for practical applications, either. We hope to be able to discuss these and other questions related to the problem of (dis)belief change in future work.

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