Partial Logic and the Dynamics of Epistemic States

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0. Introduction

This paper proposes a new approach for the modelization of the dynamics of epistemic states using *partial* logic. It has three parts. In the first one, we present in a nutshell the classical Alchourrón-Gärdenfors-Makinson (AGM) dynamics of epistemic states. We put forward a very intuitive property that any epistemic state should possess after a contraction on B (to not contain $A \supset B$ if it does not contain $\neg A$) and show that the AGM approach does not satisfy this property but for trivial cases. We suggest that this problem is related to the use of classical propositional logic.

In the second part, we briefly present a partial semantics using three truth values: the true, the false and the undefined. Two very general constraints are introduced, that of *monotonicity* of truth functions and that of *maximality*. A complete system is provided, and it is shown by a "representation" theorem that it is the only one that satisfies the two constraints.

In the third and last part, we show that using these $\vee$-saturated sets as epistemic states (with an appropriate definition of expansion and contraction), we can define a dynamics of epistemic states that satisfies all except two of the AGM postulates and that also satisfies the property stated above. The bulk of the proofs are given in the addendum.

1. One problem with AGM

According to AGM, an epistemic state for a propositional language L, is a set K of propositions closed under the relation of logical consequence, i.e., $A \in K$ iff $K \vdash A$ where $\vdash$ is at least as strong as the relation of logical consequence in classical propositional logic. A proposition $A \in L$ is said to be accepted (believed) iff $A \in K$, rejected iff $\neg A \in K$, and undetermined iff $A \notin K$ and $\neg A \notin K$.

1.1 Expansion

The expansion of an epistemic state K by A is the logical closure of the set obtained by adding A to K. Formally $K_A^+ = Cn(K \cup \{A\})$.

Expansion functions satisfy the following postulates:

(K_+1) K_A^+ is an epistemic state
(K_+2) $A \in K_A^+$
(K_+3) $K \subseteq K_A^+$
(K_+4) If $A \in K$, then $K = K_A^+$
(K_+5) If $K \subseteq H$, then $K_A^+ \subseteq H_A^+$
(K_+6) K_A^+ is the smallest set satisfying (K_+1) - (K_+5).

We have the following representation theorem [Gär 88, p.51].

Proposition 1: An expansion function satisfies (K_+1) - (K_+6) iff $K_A^+ = Cn(K \cup \{A\})$.

1.2 Contraction

The contraction of an epistemic state K consists in forming a new epistemic state K_A^- which does not contain A but is minimally different from K. The postulates are:

(K_-1) K_A^- is an epistemic state
(K_-2) $K_A^- \subseteq K$
(K_-3) If $A \notin K$, then $K_A^- = K$
(K_-4) If $\nvdash A$, then $A \notin K_A^-$
(K_-5) If $A \in K$, then $K \subseteq (K_A^-)_A^+$
(K_-6) If $\vdash (A \equiv B)$, then $K_A^- = K_B^-$.

In order to provide a representation theorem, Gärdenfors introduces the notion of a A-maximal subset.

Definition: K' is a A-maximal subset of K, iff

(i) $K' \subseteq K$
(ii) $A \notin Cn(K')$
(iii) For any K'' such that $K \subset K'' \subseteq$ K, $A \in Cn(K'')$ (or, equivalently, for any $B \in K$, if $B \notin K'$ then $(B \supset A) \in K'$).

Gärdenfors then shows that any A-maximal set satisfying (K^-1)-(K^-6), also satisfies

(K^-F) If $B \in K$ and $B \notin K^-_A$, then $(B \supset A) \in K^-_A$.

The converse is the representation theorem [Gär 88, p.77]:

Proposition 2: Any contraction function satisfying (K^-1)-(K^-6) and (K^-F) is a function that selects a maximal subset.

Taking maximal subsets as epistemic states after contraction raises a problem. One can easily show the following.

Proposition 3: For any epistemic state K, any A-maximal subset K' and any B, if $\in K$, $(B \supset A) \in K'$ or $(\neg B \supset A) \in K'$.

If this property seems quite natural when $\neg B \in K'$ or $B \in K'$, it is barely acceptable when $\neg B \notin K'$ and $B \notin K'$. Suppose, for example, that an agent believes that it is raining in Montréal (A) and has no idea if it is raining in New York (B). Suppose further that this agent ceases to believe A. According to AGM, this agent still believes either $(B \supset A)$ or $(\neg B \supset A)$. Both are unacceptable because should this agent come to believe that B or that $\neg B$, he will again believe that A. AGM were aware of that difficulty [Alc 82].

A first attempt to solve it was to take as epistemic states the intersection of all the A-maximal sets (*full meet contraction functions*). But this solution is even worse because we can show that this intersection contains all and only the consequences of $\neg A$ which belong to K.

Proposition 4: For any K and any $A \in K$, if we define $K^-_A = \bigcap_i X_i$ where the X_i are the A-maximal subsets, then $B \in K^-_A$ iff B $\in K \cap Cn(\neg A)$.

There is in fact a very large variety of A-maximal subsets of K as it is shown by the following lemmas.

Lemma 1: Let K be an epistemic state, $A \in K$ and $Y \subseteq K$ be such that $Y \nvdash A$. There is then a A-maximal subset X of K such that $Y \subseteq X$.

Lemma 2: If $A \in K$, $\nvdash A$ and $\nvdash (B \supset A)$ and $\nvdash (\neg B \supset A)$, then there is a A-maximal $X \subseteq K$ such that $(\neg B \supset A) \in X$ and a A-maximal $Y \subseteq K$ such that $(B \supset A) \in Y$.

The solution retained by Gärdenfors is that of *partial meet contraction functions*, i.e., to define $K^-_A = \bigcap_i X_i$ where the X_i are some but not all of the maximal subsets. We have the following representation theorem [Gär 88, p.80].

Proposition 5: A contraction function is a partial meet contraction function iff it satisfies ($K_$1)-($K_$6).

Independently of the problem concerning the selection of the maximal states or the question of the epistemic entrenchment of the sentences in K, this approach is barely acceptable. Let's take a K such that $A \in K$, $B \notin K$ or $\neg B \notin K$. We have seen that each A-maximal subset will contain $(\neg B \supset A)$ or $(B \supset A)$, but there is no reason to have $(\neg B \supset A) \in K^-_A$ or $(B \supset A) \in K^-_A$ when there is no logical, semantical or causal link between A and B. More precisely, the following principle seems to be quite natural.

Principle of cautiousness: If an agent believes $(B \supset A)$ on the sole basis that he believes A (i.e., he does not believe $\neg B$), then if he ceases to believe A, he ceases to believe $(B \supset A)$.

Or, to put it in disjunctive form: If an agent believes $(B \vee A)$ on the sole basis that he believes A (i.e., he does not believe B), then if he ceases to believe A, he ceases to believe $(B \vee A)$.

In order to get rid of all these undesirable implications where the antecedent is undetermined, we have to find for any A-maximal X_i containing $(B \supset A)$ a A-maximal X_j containing $(\neg B \supset A)$ and then both implications disappear when we take $X_i \cap X_j$. But it is hard to see how this can be done but on an *ad hoc* basis, because, when A is true, classical logic cannot make a difference for $(B \supset A)$ between the case where B is false and that where B is undetermined.

This can be compared with the case where $\neg B \in K$. In that case we also have that $(B \supset A) \in K$ and $(\neg B \supset A) \in K$, but then, in order to preserve consistency, the agent must chose between dropping $(\neg B \supset A)$ on one side or $\neg B$ and $(B \supset A)$ on the other side, and this can be done in the light of extra logical reasons. For example, if the agent strongly believes in the existence of a counterfactual link between B and A, he will choose to drop $(\neg B \supset A)$; if he believes that there is no link between B and A, he will choose to drop $\neg B$ and $(B \supset A)$. But when neither B nor $\neg B$ are accepted and there is no link between A and B, it seems natural, even in minimal changes, to drop both $(B \supset A)$ and $(\neg B \supset A)$ even if there is no consistency constraint to do so. The notion of partial meet contraction function does not seem to capture this simple idea because there is absolutely no room in classical logic for undetermination: a disjunction may be true even if both of the disjuncts are undetermined.

A trivial solution would be to force epistemic states to be $\vee$-saturated: if an agent has no opinion on A and no opinion on B, then he has no opinion on $(B \vee A)$. But then, once more, because the logic is classical, every epistemic state contains $(\neg A \vee A)$ and so would be a *maximally consistent set*, i.e., each agent will be omniscient. There is a way out: to drop classical logic and go partial.

2. An outline of a partial propositional logic

2.1 PARTIAL INTERPRETATIONS

Let L be the language of classical propositional logic, i.e.,

(i) For any $n \in \omega$, $p_n \in L$
(ii) If $A, B \in L$, $\neg A \in L$ and $(A \wedge B) \in L$
(iii) Nothing else belongs to L.

The other connectives are introduced as usual. We define a partial interpretation for L.

$I : L \rightarrow \{0, 1, \perp\}$ (= **3**)

$I(p_n)$ is any element of **3**

$I(\neg A) = 1$ if $I(A) = 0$
$I(\neg A) = 0$ if $I(A) = 1$
$I(\neg A) = \perp$ if $I(A) = \perp$

$I(A \wedge B) = 1$ if $I(A) = 1$ and $I(B) = 1$

$I(A \wedge B) = 0$ if $I(A) = 0$ or $I(B) = 0$
$I(A \wedge B) = \bot$ otherwise.

The values of other connectives are also those of Kleene's strong logic.

The set $\{0, 1, \bot\}$ is naturally partially ordered in the following way.

$0 \sqsubseteq 0,\ 1 \sqsubseteq 1,\ \bot \sqsubseteq \bot,\ \bot \sqsubseteq 0,\ \bot \sqsubseteq 1.$

According to this order, all the definable connectives are monotonic. From the point of view of epistemic logic, this property of monotonicity is very interesting because it authorizes to interprete $\bot$ as meaning "undefined". For example, according to the truth table given above, "$I(A \wedge B) = 1$ if $I(A) = 1$ and $I(B) = 1$", can be interpreted as saying that it is necessary that A and B be true to conclude that $(A \wedge B)$ is true. In the same way, "$I(A \wedge B) = 0$ if $I(A) = 0$ or $I(B) = 0$", can be interpreted as saying that it is a sufficient reason that A be false or B be false to conclude that $(A \wedge B)$ is false. Finally, "$I(A \wedge B) = \bot$ otherwise", can be interpreted as saying that we cannot conclude in the other cases.

It can be shown that all the truth functions definable using the set $\{\neg, \wedge\}$ are monotonic.[1]

Functions definable using the set $\{\neg, \wedge\}$ are not only monotonic, but are also *maximal* in the following sense.Let $f, g \in 3^{(3^n)}$ (f,$g$are not necessarily monotonic). Let us define $f \sqsubseteq g$ iff for any $<x_0, ..., x_{n-1}> \in 3^n$, $f(<x_0, ..., x_{n-1}>) \sqsubseteq g(<x_0, ..., x_{n-1}>$. Now, let f be monotonic. Then,f is *maximal* iff for any $g \in 3^{(3^n)}$, $g \neq f$, if $f \sqsubseteq g$ then g is *not* monotonic.
We will need the following definitions and propositions.

Definition: Let I, I' be two partial interpretations. We will say that $I \sqsubseteq I'$ iff, for any propositional symbol p_n, $I(p_n) \sqsubseteq I'(p_n)$.

Proposition 6: $I \sqsubseteq I'$ iff for any $A \in L$, $I(A) \sqsubseteq I'(A)$.

Definition: A partial interpretation I is *total* iff for any p_n, $I(p_n) \neq \bot$.

Proposition 7: I is total iff for any A, $I(A) \neq \bot$.

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[1] In fact, if **V** and ★ (the name of 1 and of $\bot$ respectively) are added to L, $\{\neg, \wedge\}$ is functionally complete for monotonic functions. See [Thi 92].

Definition: A formula A is t-valid iff for any I, $I(A) = 1$.

Definition: A formula A is f-valid iff for any I, $I(A) \neq 0$.

Proposition 8: There is no t-valid formula and the set of f-valid formulas is the set of classical tautologies.

Definition: Let $\Gamma \subseteq L$ and $A \in L$, A is a t-valid consequence of Γ iff for any I, if $I(X) = 1$ for any $X \in \Gamma$, then $I(A) = 1$.

2.2 A SYSTEM FOR PARTIAL PROPOSITIONAL LOGIC

The following system is a variation of [Thi 92].

Definition: A derivation of A from Γ is a finite sequence of formulas such that the last one is A, each member of the sequence is either a member of Γ or is obtained by the application of one the following rules on previous formulas of the sequence.

R1. $A \wedge \neg A \vdash B$
R2. $\neg\neg A \vdash A$
R3. $A \vdash \neg\neg A$
R4. $A \wedge B \vdash A$
R5. $A \wedge B \vdash B$
R6. $A \vdash \neg(\neg A \wedge \neg B)$
R7. $A \vdash \neg(\neg B \wedge \neg A)$
R8. $\neg(\neg\neg B \wedge \neg\neg A) \vdash \neg(A \wedge B)$
R9. $\neg(A \wedge B) \vdash \neg(\neg\neg B \wedge \neg\neg A)$
R10. If $\Gamma \vdash A$ and $\Gamma \vdash B$, then $\Gamma \vdash A \wedge B$
R11. If $\Gamma, A \vdash C$ and $\Gamma, B \vdash C$, then $\Gamma, \neg(\neg A \wedge \neg B) \vdash C$

Proposition 9: The preceding set of rules is sound and strongly complete for partial interpretations (according to t-validity).

The proof of soundness is straightforward. The proof of completeness goes along the following lines.

Proposition 10: Any consistent set Δ of formulas such that for some A, $\Delta \not\vdash A$, can be embedded into a set Γ such that
(i) $\Gamma \not\vdash A$
(ii) Γ is deductively closed for $\vdash$
(iii) Γ is $\vee$-saturated (i.e., if $\neg(\neg A \wedge \neg B) \in \Gamma$, then $A \in \Gamma$ or $B \in \Gamma$).

Because logic is partial, it does not follow from (i), (ii) and (iii) that Γ is *maximally* consistent. For example, if $A \notin \Gamma$ and $\neg A \notin \Gamma$, then $\neg(\neg A \wedge A) \notin \Gamma$.

Proposition 11: For any Γ satisfying the properties above, there is an interpretation I_Γ such that, for any $A \in L$:

(i) $I_\Gamma(A) = 1$ if $A \in \Gamma$
(ii) $I_\Gamma(A) = 0$ if $\neg A \in \Gamma$
(iii) $I_\Gamma(A) = \bot$ if $A \notin \Gamma$ and $B \notin \Gamma$.

Strong completeness follows as usual. This completeness proof gives rise to the following "representation" theorem. Suppose you have two systems of partial logic *S* and *S*', both of them satisfying the following property:

> Every non empty closed saturated subset of *L* defines values of $\neg$, $\wedge$ which are *monotonic* and *maximal*

then *S* and *S*' are equivalent logic.

We will now use these $\vee$-saturated sets as epistemic states.

3. Partial interpretations as epistemic states

Definition: An epistemic state is a partial interpretation or, equivalently, because of completeness, a consistent $\vee$-saturated set closed under $\vdash$.

3.1 EXPANSION

Definition: Let Γ be an epistemic state. An expansion Γ_A^+ is a a deductively closed $\vee$-saturated set of $\Gamma \cup \{A\}$. (If $\Gamma \cup \{A\}$ is inconsistent, Γ_A^+ is L).

Clearly, Γ^+_A is not unique. We postulate the existence of a *selection function* that selects a particular enumeration of the elements of L. Using this particular enumeration in defining the superset of $\Gamma \cup \{A\}$, *à la* Lindenbaum, we will have unicity.

One can easily show that the following postulates are satisfied.

Proposition 12:

(I_+1) Γ^+_A is an epistemic state
(I_+2) $A \in \Gamma^+_A$
(I_+3) $\Gamma \subseteq \Gamma^+_A$
(I_+4) If $A \in \Gamma$, then $\Gamma = \Gamma^+_A$
(I_+5) Γ^+_A is the smallest set satisfying (I_+1) - (I_+4).

Only one of the AGM postulates has no equivalent: (K_+5) If $K \subseteq H$, then $K^+_A \subseteq H^+_A$. This is a consequence of $\vee$-saturation. For example,

$\varnothing \subseteq \varnothing^+_A$ and $\varnothing \subseteq \varnothing^+_{\neg A}$
but
$\varnothing_A{}^+_{\vee\neg A} \not\subseteq \varnothing^+_A{}_{A\vee}{}^+_{\neg A}$ or $\varnothing_A{}^+_{\vee\neg A} \not\subseteq \varnothing^+_{\neg A}{}_{A\vee}{}^+_{\neg A}$.

3.2 CONTRACTION

Definition: Let Γ be a $\vee$-saturated set. Any $\vee$-saturated subset $X \subseteq \Gamma$ such that $X \nvdash A$ is said to be a A-$\vee$-saturated subset of Γ.

Definition: $X \subseteq \Gamma$ is a *maximal* A-$\vee$-saturated subset of Γ iff for any $B \in \Gamma - X$, either $X \cup \{B\} \vdash A$ or $X^+_B \not\subseteq \Gamma$.

Proposition 13: For any Γ and any A, there is a maximal A-$\vee$-saturated subset of Γ.

In general, there are many maximal A-$\vee$-saturated subsets of Γ. These can be defined in the same way we used in section 2. Starting with $\varnothing$ (which is a A-$\vee$-saturated subset of Γ) and any enumeration $(A_i) = A_0, A_1, \ldots, A_n, \ldots$ of the members of Γ, we defined the sequence $\Gamma_0, \ldots, \Gamma_n, \ldots$ of A-$\vee$-saturated subsets of Γ in the following way.

$\Gamma_0 = \emptyset$.

Induction step. If there is a A-$\vee$-saturated subset X_n of Γ such that $(\Gamma_{n-1} \cup \{A_{n-1}\}) \subseteq X_n$, then $\Gamma_n = X_n$, otherwise $\Gamma_n = \Gamma_{n-1}$. One can easily show that $\bigcup_i \Gamma_i$ is a maximal A-$\vee$-saturated subset of Γ.

If, in the induction step, the A-$\vee$-saturated subset of Γ is defined using Lindenbaum's lemma with the members of (A_i) in the order given by the selection function and A is the test-formula for consistency, then we obtain a *unique* maximal A-$\vee$-saturated subset of Γ. We note it Γ^-_A. The following postulates are satisfied:

Proposition 14:

(I_-1) Γ^-_A is an epistemic state
(I_-2) $\Gamma^-_A \subseteq \Gamma$
(I_-3) If $A \notin \Gamma$, then $\Gamma^-_A = \Gamma$
(I_-4) $A \notin \Gamma^-_A$
(I_-5) If $A \vdash B$ and $B \vdash A$, then $\Gamma^-_A = \Gamma^-_B$.

Again, there is one AGM postulate that is not satisfied

(K_-5) If $A \in K$, then $K \subseteq (K^-_A)^+_A$

and again it is a consequence of $\vee$-saturation.

Proposition 15: Any contraction function satisfies the *Principle of cautiousness*, i.e., if $B \notin \Gamma$, then $(B \vee A) \notin \Gamma^-_A$.

Three final remarks. Firstly, if we define a revision function using Levi's identity, i.e., $\Gamma^*_A = \Gamma^{-\,+}_{\neg A A}$, the following postulates are satisfied.

Proposition 16:

(I_*1) Γ^*_A is an epistemic state
(I_*2) $A \in \Gamma^*_A$
(I_*3) $\Gamma^*_A \subseteq \Gamma^+_A$
(I_*4) If $\Gamma \cup \{A\}$ is consistent, $\Gamma^+_A \subseteq \Gamma^*_A$
(I_*5) If Γ is consistent, then Γ^*_A is inconsistent iff $\{A\}$ is inconsistent.

(I_*6) If $A \vdash B$ and $B \vdash A$, then $\Gamma^*_A = \Gamma^*_B$.

Secondly, it may seem counterintuitive to suppose that any rational agent who has an opinion on $(A \supset A)$ or equivalently on $(\neg A \vee A)$, must also have one on A and on $\neg A$. But there is no damage because the only differences between the consequences of Γ and the consequences of $\Gamma \cup \{(A \supset A)\}$ are classical tautological consequences of Γ. It is however possible to modelize the fact that an agent gives some credit to A and some to $\neg A$. Let's suppose that $(\neg A \vee A) \notin \Gamma$. If neither A nor $\neg A$ are tautologies, we have $\vee$-saturated supersets of both $\Gamma \cup \{A\}$ and $\Gamma \cup \{\neg A\}$. Then, the hesitation of the agent can be represented by a probability distribution on saturated sets.

Thirdly, and this is an important point, this technique for providing partial interpretations can be straightforwardly extended to first-order predicate logic and to finite type theory for which a corresponding representation theorem holds.

Addendum

Proposition 3: For any epistemic state K, any A-maximal subset K' and any B, if $A \in K$, $(B \supset A) \in K'$ or $(\neg B \supset A) \in K'$.

Proof

Let K' be a A-maximal subset of K and let B be any sentence. There are two cases.

(1) $B \in K$. Because $\vdash (A \supset (B \supset A))$ and $A \in K$, by *modus ponens*, $(B \supset A) \in K$. Because K' is A-maximal, if $(B \supset A) \notin K'$, then $((B \supset A) \supset A) \in K'$. But $((B \supset A) \supset A)$ is tautologicaly equivalent to $(\neg B \supset A)$, and thus $(\neg B \supset A) \in K'$.

(2) $B \notin K$. In that case $(B \supset A) \in K$ and $(\neg B \supset A) \in K$ because $\vdash (A \supset (B \supset A))$ and $\vdash (A \supset (\neg B \supset A))$. The case is similar to (1).

Lemma 1: Let K be an epistemic state, $A \in K$ and $Y \subseteq K$ be such that $Y \nvdash A$. There is then a A-maximal subset X of K such that $Y \subseteq X$.

Proof

Let $A_0, \ldots, A_n, \ldots$ be an enumeration of the elements of K. Let us now consider the following sequence of sets.

$X_0 = Y$

If $X_i \cup \{A_i\} \nvdash A$, then $X_{i+1} = X_i \cup \{A_i\}$

If $X_i \cup \{A_i\} \vdash A$, then $X_{i+1} = X_i \cup \{A_i \supset A\}$.

We show that $X = \bigcup_i X_i$ is A-maximal subset of K such that $Y \subseteq X$.
Claim: For any i, $X_i \nvdash A$.

$X_0 = Y \nvdash A$ (hypothesis).

Let us suppose that $X_i \nvdash A$.

If $X_i \cup \{A_i\} \nvdash A$, then $X_{i+1} = X_i \cup \{A_i\} \nvdash A$.

If $X_i \cup \{A_i\} \vdash A$, then $X_{i+1} = X_i \cup \{A_i \supset A\}$.
Let us suppose that $X_i \cup \{A_i \supset A\} \vdash A$. By the deduction theorem, $X_i \vdash ((A_i \supset A) \supset A)$. But $((A_i \supset A) \supset A)$ is tautologically equivalent to $(\neg A_i \supset A)$ and thus $X_i \vdash (\neg A_i \supset A)$. By hypothesis, $X_i \cup \{A_i\} \vdash A$ and so by the deduction theorem, $X_i \vdash (A_i \supset A)$. Finally, because $\vdash ((A_i \supset A) \supset ((\neg A_i \supset A) \supset A)))$, $X_i \vdash A$, which contradicts the hypothesis.

Lemma 2: If $A \in K$, $\nvdash A$ and $\nvdash (B \supset A)$ and $\nvdash (\neg B \supset A)$, then there is a A-maximal $X \subseteq K$ such that $(\neg B \supset A) \in X$ and a A-maximal $Y \subseteq K$ such that $(B \supset A) \in Y$.

Proof
$A \in K$ implies that $(\neg B \supset A) \in K$ and $(\neg B \supset A) \in K$.
It is not the case that $\{(\neg B \supset A)\} \vdash A$ because if so, we would have $\vdash ((\neg B \supset A) \supset A)$ and thus $\vdash (B \supset A)$. So, by lemma 1, there is a A-maximal X such that $\{(\neg B \supset A)\} \subseteq X$. Similarly, it is not the case that $\{(B \supset A)\} \vdash A$ because if so, we would have $\vdash ((B \supset A) \supset A)$ and thus $\vdash (\neg B \supset A)$. So, by lemma 1, there is a A -maximal Y such that $\{(B \supset A)\} \subseteq Y$.

Proposition 8: There is no t-valid formula and the set of f-valid formulas is the set of classical tautologies.

Proof: One can easily check that if $I(p_n) = \perp$ for every n, then $I(A) = \perp$ for every A.
Now, if A is a tautology, $I(A) \neq 0$ because if $I(A) = 0$, there is an I' such that $I'(p_n) = I(p_n)$ when $I(p_n) \neq \perp$ and $I'(p_n) = 1$ otherwise. Clearly, I' is total, $I \sqsubseteq I'$, $I(A) \sqsubseteq I(A)$ and finally, $0 \sqsubseteq 1$.

Conversely, if $I(A) \neq 0$ for any I, $I'(A) \neq 0$ for any I' total. Thus $I'(A) = 1$ and A is a classical tautology.

Proposition 10: Any consistent set Δ of formulas such that for some A, $\Delta \nvdash A$, can be embedded into a set Γ such that
(i) $\Gamma \nvdash A$
(ii) Γ is deductively closed for $\vdash$
(iii) Γ is $\vee$-saturated (i.e., if $\neg(\neg A \wedge \neg B) \in \Gamma$, then $A \in \Gamma$ or $B \in \Gamma$).

Proof
Let Δ and A' be such that $\Delta \nvdash A$, and let $A_0,...,A_n,...$ be an enumeration of the elements of L such that each formula of L occurs denumerably many times. We define the following sequence of sets of formulas.
$\Gamma_0 = \Delta$
If $\Gamma_{2n} \nvdash A_n$, then $\Gamma_{2n+1} = \Gamma_{2n+2} = \Gamma_{2n}$
If $\Gamma_{2n} \vdash A_n$, then $\Gamma_{2n+1} = \Gamma_{2n} \cup \{A_n\}$ and
(1) if A_n is not $\neg(\neg B \wedge \neg C)$, then
$\Gamma_{2n+2} = \Gamma_{2n+1}$;
(2) if A_n is $\neg(\neg B \wedge \neg C)$ for some B and some C, then $\Gamma_{2n+2} = \Gamma_{2n+1} \cup \{B\}$ if $\Gamma_{2n+1} \cup \{B\} \nvdash A$ and otherwise $\Gamma_{2n+2} = \Gamma_{2n+1} \cup \{C\}$.

Claim: $\Gamma = \bigcup_i \Gamma_i$ has the properties (i)-(iii).
We show that, for any i, $\Gamma_i \nvdash A$.

$\Gamma_0 = \Delta \nvdash A$ (hypothesis).
If $\Gamma_{2n} \nvdash A_n$, then $\Gamma_{2n+1} = \Gamma_{2n+2} = \Gamma_{2n} \nvdash A$ (induction hypothesis).
If $\Gamma_{2n} \vdash A_n$, then $\Gamma_{2n+1} = \Gamma_{2n} \cup \{A_n\}$. But if $\Gamma_{2n} \cup \{A_n\} \vdash A$ and $\Gamma_{2n} \vdash A_n$, $\Gamma_{2n} \vdash A$. Thus $\Gamma_{2n+1} \nvdash A$ (induction hypothesis).
(1) If A_n is not $\neg(\neg B \wedge \neg C)$, then $\Gamma_{2n+2} = \Gamma_{2n+1} \nvdash A$.
(2) If A_n is $\neg(\neg B \wedge \neg C)$ for some B and some C, then $\Gamma_{2n+2} = \Gamma_{2n+1} \cup \{B\}$ if $\Gamma_{2n+1} \cup \{B\} \nvdash A$ and otherwise $\Gamma_{2n+2} = \Gamma_{2n+1} \cup \{C\}$. Let us suppose that $\Gamma_{2n+1} \cup \{B\}$ if $\Gamma_{2n+1} \cup \{B\} \vdash A$ and $\Gamma_{2n+2} = \Gamma_{2n+1} \cup \{C\} \vdash A$ by R11. Thus we have $\Gamma_{2n+1} \cup \{\neg(\neg B \wedge \neg C)\} \vdash A$ and thus $\Gamma_{2n+1} \vdash A$.
So we have proved that, for any i, $\Gamma_i \nvdash A$.

By the usual argument, $\Gamma = \bigcup_i \Gamma_i \nvdash A$, and (i) is proved.

Let us suppose that $\Gamma \vdash B$ and $B \notin \Gamma$. This implies that for some Γ_i, $\Gamma_i \vdash B$. Let $2n > i$ be the smallest number such that $A_n = B$. We have $\Gamma_i \subseteq \Gamma_{2n} \vdash B$ and thus $B \in \Gamma_{2n+1} \subseteq \Gamma$ which contradicts the hypothesis. This proves (ii).

(iii) follows directly from the definition of Γ_{2n+2}.

Proposition 11: For any Γ satisfying the properties above, there is an interpretation I_Γ such that, for any $A \in L$:

(i) $I_\Gamma(A) = 1$ if $A \in \Gamma$
(ii) $I_\Gamma(A) = 0$ if $\neg A \in \Gamma$
(iii) $I_\Gamma(A) = \perp$ if $A \notin \Gamma$ and $B \notin \Gamma$.

Proof

We drop the index Γ. For any n, let $I(p_n) = 1$ if $p_n \in \Gamma$, $I(p_n) = 0$ if $\neg p_n \in \Gamma$ and $I(p_n) = \perp$ otherwise.

Let A be $\neg B$. By the general definition of an interpretation, if $I(A) = 1$, then $I(B) = 0$. By induction hypothesis, $\neg B \in \Gamma$.
If $I(A) = 0$, then $I(B) = 1$. By induction hypothesis, $B \in \Gamma$, and by R3, $\neg\neg B \in \Gamma$.

Let A be $(B \wedge C)$. If $I(B \wedge C) = 1$, by the general definition of an interpretation, $I(B) = I(C) = 1$. By induction hypothesis, $B \in \Gamma$ and $C \in \Gamma$, by R10, $(B \wedge C) \in \Gamma$.
Conversely, if $(B \wedge C) \in \Gamma$, by R4 and R5, $B \in \Gamma$ and $C \in \Gamma$, by inducton hypothesis, $I(B) = I(C) = 1$ and $I(B \wedge C) = 1$.
If $I(B \wedge C) = 0$, by the general definition of an interpretation, $I(B) = 0$ or $I(C) = 0$. Thus $\neg B \in \Gamma$ or $\neg C \in \Gamma$ and by R6 or R7, $\neg(\neg\neg B \wedge \neg\neg C) \in \Gamma$. By R8, $\neg(B \wedge C) \in \Gamma$.
Conversely, if $\neg(B \wedge C) \in \Gamma$, then by R9, $\neg(\neg\neg B \wedge \neg\neg C) \in \Gamma$. By $\vee$-saturation, $\neg\neg B \in \Gamma$ or $\neg\neg C \in \Gamma$. By R2, $B \in \Gamma$ or $C \in \Gamma$.

Proposition 15: Any contraction function satisfies the *Principle of cautiousness*, i.e., if $B \notin \Gamma$, then $(B \vee A) \notin \Gamma^-_A$.

Proof

If $B \notin \Gamma$, then $B \notin \Gamma_{\bar{A}}$. But $A \notin \Gamma_{\bar{A}}$. Thus, by $\vee$-saturation, $(B \vee A) \notin \Gamma_{\bar{A}}$.

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