A Sequent Formulation of Conditional Logic Based on Belief Change Operations[1]

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As part of his well-known investigation of the theory of belief change Peter Gärdenfors has developed a semantics for conditional logic, based on the operations of expansion and revision of states of information.[2] The account amounts to a formalisation of the Ramsey test for conditionals. A conditional $A > B$ is declared accepted in a state of information K if B is accepted in the state of information which is the result of revising K with respect to A. While Gärdenfors's account takes the truth-functional part of the logic as given, the present paper proposes a semantics entirely based on epistemic states and operations on these states. The semantics is accompanied by a syntactic treatment of conditional logic which is formally similar to Gentzen's sequent formulation of natural deduction rules.[3] The basic idea underlying the approach is inspired by Gärdenfors's proposal to interpret propositions as certain functions on epistemic states.[4]

I

The language **L** to be considered here is a propositional language with the connectives ~, &, $\vee$, $\supset$, and >, the last of these being of course the conditional forming connective. A semantics based on models is provided for the language as follows.

A model $\boldsymbol{M}$ is a quadruple $\langle \boldsymbol{K}, +, *, \ni \rangle$. $\boldsymbol{K} = <|\boldsymbol{K}|, \leq, \mathrm{V}, \Lambda, K_\perp, K_\top>$ here is a lattice whose universe $|\boldsymbol{K}|$, usually also denoted $\boldsymbol{K}$, is a set of epistemic states. $\leq$ is the lattice ordering relation, $K \mathrm{V} K'$ is the join and $K \Lambda K'$ the meet of elements K and K' of $\boldsymbol{K}$, $K_\perp$ is the unit element and $K_\top$ the null element. $K \leq K'$ signifies that information K is included in K'; $K \mathrm{V} K'$ can be interpreted as the state of information obtained by combining K and K'; $K \Lambda K'$ as the information common to K and K'; $K_\top$ as a priori knowledge, while $K_\perp$ is what is frequently called the 'absurd' state of information in which anything is believed indiscriminately.

+ and * are functions, namely expansion and revision, from $\boldsymbol{K} \times \mathbf{L}$ to $\boldsymbol{K}$. The expansion and the revision of K with respect to A are written $(K)+(A)$ and $(K)*(A)$, respectively, with brackets omitted where possible. $+A$ and $*B$ then indicate operations on epistemic states. Greek letters α, β, ... stand for (possibly empty) sequences of such epistemic operations, $K\alpha$ being the result of applying to K the first, then the second, etc. of these operations. $\ni$ is a

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[1] I am grateful to Lloyd Humberstone for alerting me to errors and for very helpful suggestions.

[2] (Gärdenfors 1988), Ch. 7.

[3] The only similar treatment I am aware of is a Fitch-style formulation of **VWS** in (Thomason 1970).

[4] (Gärdenfors 1988), Ch. 6.

relation on $\boldsymbol{K} \times \mathbf{L}$; and $K \ni A$ signifies that statement A is accepted in epistemic state K. The functions and the relation are subject to constraints from the following list.

(1.1) *If* $K \ni A$ *then, for every* $K' \geq K$, $K' \ni A$

(1.2) *If* $K \ni A$ *and* $K' \ni A$, *then* $(K \Lambda K') \ni A$

(1.3) *If, for every* K', $K \leq K' \neq K_{\perp}$, *there exists* K'', $K' \leq K'' \neq K_{\perp}$, *such that* $K'' \ni A$, *then* $K \ni A$

(2) $K \ni {\sim}A$ *iff, for every* K', $K \leq K' \neq K_{\perp}$, not: $K' \ni A$

(3) $K \ni A \,\&\, B$ *iff* $K \ni A$ *and* $K \ni B$

(4) $K \ni A \vee B$ *iff* $K{+}A \;\Lambda\; K{+}B \leq K$

(5) $K \ni A \supset B$ *iff* $K{+}A \ni B$

(6) $K \ni A > B$ *iff* $K{*}A \ni B$

(+1) $K{+}A \ni A$

(+2) $K \leq K{+}A$

(+3) *If* $K \ni A$, then $K{+}A \leq K$

(*1) $K{*}A \ni A$

(*2) *If* $K{*}A \ni B$ *and* $K{*}B \ni A$, *then* $K{*}A = K{*}B$

(*3) $K{*}(A \,\&\, B) \leq K{*}A{+}B$

(*4) *If for every* K', $K \leq K' \neq K_{\perp}$, $K'{*}A{+}B \neq K_{\perp}$, *then* $K{*}A{+}B \leq K{*}(A \,\&\, B)$

(*W) $K{*}A \leq K{+}A$

(*C) *If* $K \ni A$, *then* $K \leq K{*}A$

(*S) *If for every* K', $K \leq K' \neq K_{\perp}$, $K'{*}A{+}B \neq K_{\perp}$, *then* $K{*}A \ni B$

Several of these constraints correspond to constraints standardly adopted for expansion and contraction of belief states. **(+1)**, **(+2)** and **(+3)** correspond to K+2, K+3, and K+4; **(*1)**, **(*3)**, **(*4)**, and **(*W)** to K*2, K*7, K*L and K*3; **(*2)** and **(*C)** are related to K*6 and K*4w.[5]

We consider three kinds of models distinguished by the sets of constraints satisfied in each case. If constraints **(1.1)–(6)**, **(+1)–(+3)**, **(*1)–(*4)** and **(*W)** are met, the model is a ***VW**-model*. If **(*C)** is also met, the model is a ***VC**-model*. And a ***VCS**–model* is a **VC**–model in which constraint **(*S)** is satisfied.[6] Intuitionistic versions of **VW**- and **VC**-models can be obtained by deleting constraint **(1.3)**. There is no intuitionistic version of **VCS**-models. Constraint **(1.3)** is implied by the remaining **VCS**-constraints.

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[5] See (Gärdenfors 1988), Ch. 3 and Ch. 7.

[6] **(*4)** can then be dispensed with.

Having introduced models it is possible to define a semantic notion of logical consequence in the system **VW**:

$$\Gamma \models A$$
if and only if
for every **VW**-*model* $\boldsymbol{M} = \langle \boldsymbol{K}, +, *, \ni \rangle$ *and every* $K \in \boldsymbol{K}$, *if* $K \ni C$ *for every* $C \in \Gamma$,
then $K \ni A$.

Analogous definitions are given for systems **VC** and **VCS** and for the intuitionist systems paralleling **VW** and **VC**. As the labels **VW**, **VC**, and **VCS** suggest, we are dealing here with David Lewis's systems of conditional logic, a claim which will be justified later.

II

In order to define a syntactic notion of consequence we introduce the notion of a *sequent* $\alpha : A$. As before, α is a sequence of expressions for expansion and revision operations. Given a state of information K, the sequent $\alpha : A$ expresses the claim that $K\alpha \ni A$. A sequent is *provable* if it is a Basic Sequent or obtainable from Basic Sequents by means of Transformation Rules. Depending on the set of transformation rules admitted, we get different classes of provable sequents and different notions of syntactic consequence.

A sequent is said to be *provable in **VW*** if all the transformation rules except (C) and (S) are given, *provable in **VC*** if all the transformation rules except (S) are available, and *provable in **VCS*** if all transformation rules are present.[7] The two intuitionistic notions of provability do not of course use the rule (DN).

For each of these systems syntactic consequence is then defined in terms of the provability of sequents.

$$\Gamma \vdash B$$
if and only if
there exists a finite subset $\{A_1, ..., A_n\}$ of Γ *such that the sequent* $+A_1 \ldots +A_n : B$ *is provable*

Basic Sequents

(*Basic*+) $\alpha +A : A$ (*Basic**) $\alpha *A : A$

[7] In the presence of (S) rules (**3) and (C) can be dispensed with.

Transformation Rules

(*Thinning*)[8] $$\frac{\beta : A}{\alpha\,\beta : A}$$

(*Weakening*) $$\frac{\alpha : B}{\alpha +A : B}$$

(*Permutation*)[9] $$\frac{\alpha +A +B\ \beta : C}{\alpha +B +A\ \beta : C}$$

(*Cut*) $$\frac{\alpha : A \qquad \alpha +A\ \beta : C}{\alpha\,\beta : C}$$

(~I) $$\frac{\alpha +A : C \,\&\, {\sim}C}{\alpha : {\sim}A}$$

(⊥E)[10] $$\frac{\alpha : C \,\&\, {\sim}C}{\alpha : B}$$

(DN) $$\frac{\alpha : {\sim}{\sim}A}{\alpha : A}$$

(&I) $$\frac{\alpha : A \qquad \alpha : B}{\alpha : A \,\&\, B}$$

(&E) $$\frac{\alpha : A \,\&\, B}{\alpha : A} \qquad \frac{\alpha : A \,\&\, B}{\alpha : B}$$

(∨I) $$\frac{\alpha : A}{\alpha : A \vee B} \qquad \frac{\alpha : B}{\alpha : A \vee B}$$

(∨E) $$\frac{\alpha +A : C \qquad \alpha +B : C \qquad \alpha : A \vee B}{\alpha : C}$$

(⊃I) $$\frac{\alpha +A : B}{\alpha : A \supset B}$$

(⊃E) $$\frac{\alpha : A \supset B}{\alpha +A : B}$$

(>I) $$\frac{\alpha *A : B}{\alpha : A > B}$$

(>E) $$\frac{\alpha : A > B}{\alpha *A : B}$$

(**1) $$\frac{\alpha *A : B \qquad \alpha *B : A \qquad \alpha *A : C}{\alpha *B : C}$$

(W) $$\frac{\alpha *A : C}{\alpha +A : C}$$

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[8] *Thinning* need not be assumed as a primitive transformation rule, since adding operations on the left of a basic sequent yields another basic sequent and none of the transformation rules, apart from *Thinning* itself, affects such additions.

[9] Permutation of *-prefixed formulas is not in general possible. Consider as an example the epistemic states $K{*}A{*}{\sim}A$ and $K{*}{\sim}A{*}A$, which, by (*1), support ~*A* and *A*, respectively, and must therefore be different states, unless they happen to be both absurd.

[10] (⊥E) is redundant in the presence of (DN).

(**2) $$\frac{\alpha *(A \,\&\, B) : C}{\alpha *A +B : C}$$

(C) $$\frac{\alpha +A : C}{\alpha +A *A : C}$$

(**3) $$\frac{\alpha *A +B : C \qquad \alpha : {\sim}(A > {\sim}B)}{\alpha *(A \,\&\, B) : C}$$

(S) $$\frac{\alpha : {\sim}(A > {\sim}B)}{\alpha *A : B}$$

III

In order to prove soundness, it is established first that if a sequent $\alpha : C$ is provable in a system, then for every model $\boldsymbol{M} = \langle \boldsymbol{K}, +, *, \ni \rangle$ of that system and every state of information $K \in \boldsymbol{K}$, $K\alpha \ni C$. It suffices to show that the claim is true for basic sequents and that the transformation rules preserve the feature in question.

(*Basic*+) By **(+1)** $K{+}A \ni A$.

(*Basic**) By **(*1)** $K{*}A \ni A$.

(*Thinning*) Suppose that, for every K, $K\beta \ni C$. Then for every K', $K'\alpha\beta \ni C$.

(*Weakening*) $K\alpha \leq K\alpha{+}A$ by **(+2)**. Hence if $K\alpha \ni C$, $K\alpha{+}A \ni C$.

(*Permutation*) $K \leq K{+}A$ by **(+2)**. Hence $K{+}B \leq K{+}A{+}B$ by Lemma 1. $K{+}A \ni A$ by **(+1)**, hence $K{+}A{+}B \ni A$ by **(+2)** and **(1.1)**, and so $K{+}B{+}A \leq K{+}A{+}B$ by Lemma 2.

(*Cut*) Suppose $K \ni A$ and $K{+}A\beta \ni C$. By **(+2)** and **(+3)** $K{+}A = K$. Hence $K\beta \ni C$.

(&I) and (&E) By **(3)**.

(~I) Suppose $K{+}A \ni C \,\&\, {\sim}C$. Then by **(3)** $K{+}A \ni C$ and by **(2)** for every K', $K{+}A \leq K' \neq K_{\perp}$, not: $K' \ni C$. So $K{+}A = K_{\perp}$. Hence, for any K', $K \leq K' \neq K_{\perp}$, not: $K' \ni A$ by Lemma 2. So $K \ni {\sim}A$ by **(2)**.

(⊥E) If $K \ni C \,\&\, {\sim}C$, then $K = K_{\perp}$ by Lemma 3. Hence $K \ni B$ by Lemma 4.

(DN) Suppose $K \ni {\sim}{\sim}A$, i.e. by **(2)** $K' \ni {\sim}A$ for every K' such that $K \leq K' \neq K_{\perp}$, which means that for every K' such that $K \leq K' \neq K_{\perp}$, there exists K'' such that $K' \leq K'' \neq K_{\perp}$ and $K'' \ni A$. Hence $K \ni A$ by **(1.3)**.

(∨I) Suppose $K \ni A$. Then $K{+}A \leq K$ by **(+3)**. But $K{+}A \wedge K{+}B \leq K{+}A$. Hence $K{+}A \wedge K{+}B \leq K$ and $K \ni A \vee B$ by **(4)**.

(∨E) Suppose $K \ni A \vee B$, $K{+}A \ni C$ and $K{+}B \ni C$. Since $K = K{+}A \wedge K{+}B$ by **(4)**, $K \ni C$ by **(1.2)**.

(⊃I) and (⊃E) by **(5)**.

(>I) and (>E) by **(6)**.

(**1) By **(*2)**.

(**2) By **(*3)**.

(**3) Suppose $K{*}A{+}B \ni C$ and $K \ni {\sim}(A > {\sim}B)$. By Lemma 6 for every K', $K \leq K' \neq K_{\perp}$, $K'{*}A{+}B \neq K_{\perp}$. Hence by **(*4)** $K{*}A{+}B \leq K{*}A\&B$ and by **(+2)** $K{*}A \leq K{*}A\&B$.

(W) By **(*W)**.

(C) By **(+1)** and **(*C)** $K{+}A \leq K{+}A{*}A$.

(S) By **(*S)**.

So, if the sequent $+A_1\ +A_2\ \ldots\ +A_n : C$ is provable in **VW** (in **VC**, in **VCS**), then for every **VW**-model (**VC**–model, **VCS**–model) $\boldsymbol{M} = \langle \boldsymbol{K}, +, *, \ni\rangle$ and for every $K \in \boldsymbol{K}$, $K{+}A_1{+}A_2\ldots{+}A_n \ni C$. But by **(*2)** and **(*3)** $K{+}A_1{+}A_2\ldots{+}A_n = K$ if $K \ni A_1$, $K \ni A_2,\ldots$, $K \ni A_n$. Hence if sequent $+A_1\ +A_2\ \ldots\ +A_n$: C is provable in **VW** (in **VC**, in **VCS**), then for every **VW**-model (**VC**–model, **VCS**–model) $\boldsymbol{M} = \langle \boldsymbol{K}, +, *, \ni\rangle$ and for every $K \in \boldsymbol{K}$, if $K \ni A_1$, $K \ni A_2,\ldots$, $K \ni A_n$, then $K \ni C$. So, if $\{A_1, A_2, \ldots, A_n\} \subseteq \Gamma$ and sequent $+A_1\ +A_2\ \ldots\ +A_n$: C is provable in **VW** (in **VC**, in **VCS**), then for every **VW**-model (**VC**–model, **VCS**–model) $\boldsymbol{M} = \langle \boldsymbol{K}, +, *, \ni\rangle$ and for every $K \in \boldsymbol{K}$, if $K \ni B$ for every $B \in \Gamma$, then $K \ni C$. Given the definitions of $\vdash$ and $\models$ it then follows for each one of the systems **VW**, **VC** and **VCS**, as well as the two intuitionist systems, that if $\Gamma \vdash C$, then $\Gamma \models C$.

IV

Completeness of the systems **V**, **VW**, and **VC** relative to the consequence relations defined by **V**-, **VW**-, and **VC**-models, respectively, is established with the help of canonical models. The canonical model for a system is $\mathbf{M_C} = \langle \mathbf{K_C}, +_C, *_C, \ni_C\rangle$, where $\mathbf{K_C}$ is the family of sets of statements of **L** which are closed under the syntactic consequence relation of that system, where the lattice ordering is the relation of set inclusion, the lattice meet of K and K' is $K \cap K'$, the lattice join is the deductive closure of $K \cup K'$, K_T is the set of theorems, and $K_{\perp}$ is the set of all sentences of **L**, hence the only inconsistent member of $\mathbf{K_C}$. For $K \in \boldsymbol{K}_\mathbf{C}$ we put

$$K{+}_C A = \{D : A \supset D \in K\},$$

$$K{*}_C A = \{D : A > D \in K\}$$

and

$$K \ni_C A \textit{ if and only if } A \in K.$$

Note that, because of ($\supset$I) and ($\supset$E), $K{+}_C A$ is the deductive closure of $K \cup \{A\}$.

It will be shown that the canonical model defined by the consequence relation of a system is indeed a model for that system.

(a) The operations $+_C$ and $*_C$ do not lead out of $\mathbf{K_C}$; i.e. if $K \in \mathbf{K_C}$ then $K{+}_C A \in \mathbf{K_C}$ and $K{*}_C A \in \mathbf{K_C}$. Since the proofs are similar, only the proof for the second claim is presented. Suppose then that $\{B_1, ..., B_n\} \subseteq K{*}_C A$ and $\{B_1, ..., B_n\} \vdash D$. Then $\{A > B_1, \ldots, A > B_n\} \subseteq K$ by the definition of $*_C$, and the following sequents are all provable.

$$+(A > B_1) \ldots +(A > B_n) *A : B_1$$
$$\vdots$$
$$+(A > B_1) \ldots +(A > B_n) *A : B_n$$

Given that

$$+B_1 \ldots +B_n : D$$

is provable,

$$+(A > B_1) \ldots +(A > B_n) *A +B_1, \ldots, +B_n : D$$

is provable by *Thinning* and hence so is

$$+(A > B_1) \ldots +(A > B_n) *A : D$$

by repeated applications of *Cut*. By (>I)

$$+(A > B_1) \ldots +(A > B_n) : A > D$$

is then provable. Consequently $\{A > B_1, \ldots, A > B_n\} \vdash A > D$, $A > D \in K$ and $D \in K*_C A$, which means that $K*_C A$ is deductively closed and hence in $\mathbf{K_C}$.

(b) The canonical model defined by the syntactic consequence relation for a system meets the semantic constraints of that system:

(1.1) and **(1.2)** are obvious.

(1.3) Suppose $A \notin K$. Then $\sim\sim A \notin K$ by (DN). Hence $K \cup \{\sim A\}$ is consistent by (~I), the deductive closure K' of $K \cup \{\sim A\}$is in $\mathbf{K_C}$, and $K \subseteq K'$. Let K'' be any consistent deductively closed superset of K'. Then $\sim A \in K''$ and hence $A \notin K''$ by (&I).

(2) Suppose $\sim A \in K$ and $K' \supseteq K$ is consistent. Then $\sim A \in K'$ and since K' is consistent, $A \notin K'$ by (&I). On the other hand, if $\sim A \notin K$, then $K \cup \{A\}$ is consistent by (~I) and the deductive closure K' of $K \cup \{A\}$ is an element of $\mathbf{K_C}$. So there is a consistent $K' \supseteq K$ with $A \in K'$.

(3) by (&I) and (&E).

(4) First suppose $A \vee B \in K$. Assume that $D \in K+_C A$ and $D \in K+_C B$, i.e. $A \supset D \in K$ and $B \supset D \in K$. The following sequents are derivable by *Basic*+ and *Permutation*:

$$+(A \vee B) +(A \supset D) +(B \supset D) : A \vee B$$
$$+(A \vee B) +(A \supset D) +(B \supset D) +A : D$$
$$+(A \vee B) +(A \supset D) +(B \supset D) +B : D$$

Hence

$$+(A \vee B) +(A \supset D) +(B \supset D) : D$$

is derivable by (∨E). So $\{A \vee B, A \supset D, B \supset D\} \vdash D$ and $D \in K$, since K is deductively closed. Hence $K+_C A \cap K+_C B \subseteq K$. Now suppose $K+_C A \cap K+_C B \subseteq K$. $A \in K+_C A$. $A \vdash A \vee B$ by *Basic*+ and (∨I). Hence $A \vee B \in K+_C A$. Similarly, $A \vee B \in K+_C B$. Consequently, $A \vee B \in K$.

(5) $A \supset B \in K$ iff $B \in K+_{C}A$ by the definition of $K+_{C}A$.

(6) $A > B \in K$ iff $B \in K*_{C}A$ by the definition of $K*_{C}A$.

(+1) $A \supset A \in K$, since the sequent $: A \supset A$ is derivable:

$\underline{+A : A}$	(*Basic*+)
$: A \supset A$	($\supset$I)

Hence $A \in K+_{C}A$.

(+2) $K \subseteq K+_{C}A$, since $K+_{C}A$ is the deductive closure of $K \cup \{A\}$.

(+3) If $A \in K$, $K \cup \{A\} = K$ and so $K+_{C}A = K$.

(*1) $A > A \in K$, since $: A > A$ is a derivable sequent:

$\underline{*A : A}$	(*Basic**)
$: A > A$	(>I)

Hence $A \in K*_{C}A$.

(*2) By (**1).

(*3) Suppose $D \in K*_{C}(A\&B)$, i.e. $(A\&B)>D \in K$. Since one can derive

$\underline{+((A \,\&\, B) > D) *(A \,\&\, B) : D}$	(>E)
$\underline{+((A \,\&\, B) > D) *A +B : D}$	(**2)
$\underline{+((A \,\&\, B) > D) *A : B \supset D}$	($\supset$I)
$+((A \,\&\, B) > D) : A > (B \supset D)$	(>I)

$A > (B \supset D) \in K$ and hence $D \in K*_{C}A+_{C}B$.

(*4) Suppose, for every consistent $K' \supseteq K$, $K'*_{C}A+_{C}B$ is consistent. Take any consistent $K' \supseteq K$. Then $K'*_{C}A+_{C}B$ is consistent. $B \in K'*_{C}A+_{C}B$ by **(+1)**. Hence $\sim B \notin K'*_{C}A$ by **(+2)**. So $A > \sim B \notin K'$ for any consistent $K' \supseteq K$. Hence $\sim(A > \sim B) \in K$, since K is deductively closed. Now assume $D \in K*_{C}A+_{C}B$, i.e. $B \supset D \in K*_{C}A$. Then $B \supset D \in K*_{C}(A \,\&\, B)$ by (**3). But $B \in K*_{C}(A \,\&\, \mathrm{B})$ by (*1) and (3). So $\mathrm{D} \in K*_{C}(A \,\&\, \mathrm{B})$ by ($\supset$E) and Cut.

(*W) Suppose $D \in K*_{C}A$, i.e. $A > D \in K$. Then $A \supset D \in \mathrm{K}$, since one can derive in **VW**:

$\underline{+(A > B) : A > B}$	(*Basic*+)
$\underline{+(A > B) *A : B}$	(>E)
$\underline{+(A > B) +A : B}$	(W)
$+(A > B) : A \supset B$	($\supset$I)

So $D \in K+_{C}A$. Hence $K*_{C}A$.

(*C) Suppose $A \in K$. If $D \in K$, then $A > D \in K$, since one can derive in **VC**:

$\underline{+A +D : D}$	(*Basic*+)
$\underline{+D +A : D}$	(*Permutation*)
$\underline{+D +A *A : D}$	(C)
$+D +A : A > D$	(>I)

So $D \in K*_C A$ and hence $K \subseteq K+_C A$.

(*S) Suppose $\sim(A > \sim B) \in K$. Then $A > B \in K$ by (S) and (>I). So **(*S)** by Lemma 6.

Consequently, the canonical model defined by the consequence relation of any of the systems **VW**, **VC**, and **VCS** is a model for that system. Now suppose that not: $\Gamma \vdash A$. Then the closure K_Γ of Γ under consequence does not contain A. Hence there exists a model, namely the canonical model $\mathbf{M}_C = \langle \boldsymbol{K}_C, +_C, *_C, \ni_C \rangle$, and a K in $\mathbf{K}_C$, namely K_Γ, such that $K \ni_C D$ for every member D of Γ, but not: $K \ni_C A$. I.e. not: $\Gamma \models A$. So the system is complete relative to models for that system. Hence

$$\textit{If } \Gamma \models A, \textit{ then } \Gamma \vdash A,$$

this for each one of the systems **VW**, **VC** and **VCS**, and their intuitionist versions.

V

It remains to be shown that the systems **VW**, **VC** and **VCS** of conditional logic which have been characterised above are identical with the systems of the same name introduced by David Lewis.[11] I take the latter to be characterised by the axiom schemata and derivation rules used by Gärdenfors for this purpose.[12]

(A1) All truth-functional tautologies

(A2) $(A > B) \,\&\, (A > C) \supset (A > (B \,\&\, C))$

(A3) $A > (C \supset C)$

(A4) $A > A$

(A5) $(A > B) \supset (A \supset B)$

(A6) $(A \,\&\, B) \supset (A > B)$

(A7) $(A > \sim A) \supset (B > \sim A)$

(A8) $(A > B) \,\&\, (B > A) \supset ((A > C) \supset (B > C))$

(A9) $(A > C) \,\&\, (B > C) \supset ((A \vee B) > C)$

(A10) $((A > C) \,\&\, \sim(A > \sim B)) \supset ((A \,\&\, B) > C)$

(A11) $\sim(A > \sim B) \supset (A > B)$

(DR1) Modus Ponens

(DR2) If $B \supset C$ is a theorem, then $(A > B) \supset (A > C)$ is also a theorem.

[11] (Lewis 1973)

[12] (Gärdenfors 1988), Ch. 7.

VW uses (A1)–(A5) plus (A7)–(A10); **VC** uses (A1)–(A10) and **VCS** (A1)–(A11). It is not difficult to derive these axiom schemata in the corresponding calculi described above; in the case of (A4), for example, this means that every sequent of the form : $A > A$ is derivable.

Conversely, the following axiom schemata corresponding to transformation rules of the calculi described can be proved within the axiomatic systems **VW**, **VC** and **VCS**.

(**1) $((A > B) \& (B > A) \& (A > C)) \supset (B > C)$

(**2) $((A \& B) > C) \supset (A > (B \supset C))$

(**3) $((A > C) \& {\sim}(A > {\sim}B)) \supset ((A \& B) > C)$

(W) $(A > B) \supset (A \supset B)$

(C) $(A \supset B) \supset (A \supset (A > B))$

(S) ${\sim}(A > {\sim}B) \supset (A > B)$

From these the transformation rules themselves can be obtained by (DR2), the derived rule

If $B \supset C$ is a theorem, then $(A \supset B) \supset (A \supset C)$ is also a theorem,

(DR1), and the introduction- and elimination-rules for $\supset$ and $>$.

Appendix

Lemma 1. *If $K \leq K'$, then $K{+}A \leq K'{+}A$.*
Proof by **(5)**.

Lemma 2. *If $K \leq K'$ and $K' \ni A$, then $K{+}A \leq K'$.*
Proof by **(+3)** and Lemma 1.

Lemma 3. *If $K \ni C \& {\sim}C$, $K = K_\perp$.*
Proof: Suppose $K \ni C \& {\sim}C$. Then by **(3)** $K \ni C$ and by **(2)** for every K', $K \leq K' \neq K_\perp$, not: $K' \ni C$. So $K = K_\perp$.

Lemma 4. *$K_\perp \ni A$, for every A.*
Proof by **(+1)** and **(+2)**.

Lemma 5. *$K{+}A \neq K_\perp$ if and only if there exists K', $K \leq K' \neq K_\perp$, such that $K' \ni A$.*
Proof by **(+1)**, **(+2)** and **(+3)**.

Lemma 6. $K \ni \sim(A > \sim B)$ *if and only if for every* K', $K \leq K' \neq K_{\perp}$, $K'{*}A{+}B \neq K_{\perp}$.

Proof: $K \ni \sim(A > \sim B)$

iff for every K', $K \leq K' \neq K_{\perp}$, not: $K' \ni A > \sim B$ (**(2)**)

iff for every K', $K \leq K' \neq K_{\perp}$, not: $K'{*}A \ni \sim B$ (**(6)**)

iff for every K', $K \leq K' \neq K_{\perp}$, not: for every K'', $K'{*}A \leq K'' \neq K_{\perp}$, not: $K'' \ni B$ (**(2)**)

iff for every K', $K \leq K' \neq K_{\perp}$, there exists K'', $K'{*}A \leq K'' \neq K_{\perp}$, such that $K'' \ni B$

iff for every K', $K \leq K' \neq K_{\perp}$, $K'{*}A{+}B \neq K_{\perp}$ (Lemma 5)

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